8.4 Solving Radical Equations

8.4 OBJECTIVES

1. Solve an equation containing a radical expression
2. Solve an equation containing two radical expressions
3. Solve an application that involves a radical equation

In this section, we wish to establish procedures for solving equations involving radicals. The basic technique we will use involves raising both sides of an equation to some power. However, doing so requires some caution.

For example, let's begin with the equation $x = 1$. Squaring both sides gives us $x^2 = 1$, which has two solutions, 1 and -1. Clearly -1 is not a solution to the original equation. We refer to -1 as an *extraneous solution.*

NOTE

$$x^2 = 1$$
$$x^2 - 1 = 0$$
$$(x + 1)(x - 1) = 0$$

so the solutions are 1 and -1.

We must be aware of the possibility of extraneous solutions any time we raise both sides of an equation to any *even power.* Having said that, we are now prepared to introduce the power property of equality.

Rules and Properties: The Power Property of Equality

Given any two expressions a and b and any positive integer n,

If $a = b$, then $a^n = b^n$.

Note that although in applying the power property you will never lose a solution, you will often find an extraneous one as a result of raising both sides of an equation to some power. Because of this, it is very important that you *check all solutions.*

Solving a Radical Equation

NOTE Notice that

$$(\sqrt{x + 2})^2 = x + 2$$

That is why squaring both sides of the equation removes the radical.

Solve $\sqrt{x + 2} = 3$.

Squaring each side, we have

$$(\sqrt{x + 2})^2 = 3^2$$
$$x + 2 = 9$$
$$x = 7$$

Substituting 7 into the original equation, we find

$$\sqrt{7 + 2} \stackrel{?}{=} 3$$
$$\sqrt{9} \stackrel{?}{=} 3$$
$$3 = 3$$

Because this is a true statement, we have found the solution for the equation, $x = 7$.

CHECK YOURSELF 1

Solve the equation $\sqrt{x - 5} = 4$.

Example 2

Solving a Radical Equation

NOTE Applying the power property will only remove the radical if that radical is isolated on one side of the equation.

Solve $\sqrt{4x + 5} + 1 = 0$.

We must *first isolate the radical* on the left side:

$$\sqrt{4x + 5} = -1$$

NOTE Notice that on the right $(-1)^2 = 1$.

Then, squaring both sides, we have

$$(\sqrt{4x + 5})^2 = (-1)^2$$

$$4x + 5 = 1$$

and solving for x, we find that

$$x = -1$$

Now we will check the solution by substituting -1 for x in the original equation:

NOTE $\sqrt{1} = 1$, the principal root.

$$\sqrt{4(-1) + 5} + 1 \stackrel{?}{=} 0$$

$$\sqrt{1} + 1 \stackrel{?}{=} 0$$

NOTE This is clearly a false statement, so -1 is *not* a solution for the original equation.

and $2 \neq 0$

Because -1 is an extraneous solution, there are *no solutions* to the original equation.

CHECK YOURSELF 2

Solve $\sqrt{3x - 2} + 2 = 0$.

Let's consider an example in which the procedure we have described will involve squaring a binomial.

Solving a Radical Equation

NOTE These problems can also be solved graphically. With a graphing utility, plot the two graphs $Y_1 = \sqrt{x + 3}$ and $Y_2 = x + 1$. Note that the graphs have one point of intersection, where $x = 1$.

Solve $\sqrt{x + 3} = x + 1$.

We can square each side, as before.

$$(\sqrt{x + 3})^2 = (x + 1)^2$$

$$x + 3 = x^2 + 2x + 1$$

Simplifying this gives us the quadratic equation

NOTE We solved similar equations in Section 6.9.

$$x^2 + x - 2 = 0$$

Factoring, we have

$$(x - 1)(x + 2) = 0$$

which gives us the possible solutions

$x = 1 \quad \text{or} \quad x = -2$

NOTE Verify this for yourself by substituting 1 and then −2 for x in the original equation.

Now we check for extraneous solutions and find that $x = 1$ is a valid solution, but that $x = -2$ does not yield a true statement.

CAUTION

Be Careful! Sometimes (as in this example), one side of the equation contains a binomial. In that case, we must remember the middle term when we square the binomial. The square of a binomial *is always a trinomial.*

CHECK YOURSELF 3

Solve $\sqrt{x - 5} = x - 7$.

It is not always the case that one of the solutions is extraneous. We may have zero, one, or two valid solutions when we generate a quadratic from a radical equation.

In the following example we see a case in which both of the solutions derived will satisfy the equation.

Solving a Radical Equation

Solve $\sqrt{7x + 1} - 1 = 2x$.

First, *we must isolate the term involving the radical.*

NOTE Again, with a graphing utility plot $Y_1 = \sqrt{7x + 1}$ and $Y_2 = 2x + 1$. Where do they intersect?

$$\sqrt{7x + 1} = 2x + 1$$

We can now square both sides of the equation.

$$7x + 1 = 4x^2 + 4x + 1$$

Now we write the quadratic equation in standard form.

$$4x^2 - 3x = 0$$

Factoring, we have

$$x(4x - 3) = 0$$

which yields two possible solutions

$$x = 0 \quad \text{or} \quad x = \frac{3}{4}$$

Checking the solutions by substitution, we find that both values for x give true statements, as follows.

Letting x be 0, we have

$$\sqrt{7(0) + 1} - 1 \stackrel{?}{=} 2(0)$$

$$\sqrt{1} - 1 \stackrel{?}{=} 0$$

or $0 = 0$ A true statement.

Letting x be $\frac{3}{4}$, we have

$$\sqrt{7\left(\frac{3}{4}\right) + 1} - 1 \stackrel{?}{=} 2\left(\frac{3}{4}\right)$$

$$\sqrt{\frac{25}{4}} - 1 \stackrel{?}{=} \frac{3}{2}$$

$$\frac{5}{2} - 1 \stackrel{?}{=} \frac{3}{2}$$

$$\frac{3}{2} = \frac{3}{2}$$

Again a true statement.

CHECK YOURSELF 4

Solve $\sqrt{5x + 1} - 1 = 3x$.

Sometimes when an equation involves more than one radical, we must apply the power property more than once. In such a case, it is generally best to avoid having to work with two radicals on the same side of the equation. The following example illustrates one approach to the solution of such equations.

Solving an Equation Containing Two Radicals

Solve $\sqrt{x - 2} - \sqrt{2x - 6} = 1$.

First we isolate $\sqrt{x - 2}$ by adding $\sqrt{2x - 6}$ to both sides of the equation. This gives

$$\sqrt{x - 2} = 1 + \sqrt{2x - 6}$$

NOTE $1 + \sqrt{2x - 6}$ is a binomial of the form $a + b$, in which a is 1 and b is $\sqrt{2x - 6}$. The square on the right then has the form $a^2 + 2ab + b^2$.

Then squaring each side, we have

$$x - 2 = 1 + 2\sqrt{2x - 6} + 2x - 6$$

We now isolate the radical that remains on the right side.

$$-x + 3 = 2\sqrt{2x - 6}$$

We must square again to remove that radical.

$$x^2 - 6x + 9 = 4(2x - 6)$$

Now solve the quadratic equation that results.

$$x^2 - 14x + 33 = 0$$

$$(x - 3)(x - 11) = 0$$

So

$x = 3$ or $x = 11$ are the possible solutions.

Checking the possible solutions, you will find that $x = 3$ yields the only valid solution. You should verify that for yourself.

CHECK YOURSELF 5

Solve $\sqrt{x + 3} - \sqrt{2x + 4} + 1 = 0.$

Earlier in this section, we noted that extraneous roots were possible whenever we raised both sides of the equation to an *even power*. In the following example, we will raise both sides of the equation to an odd power. We will still check the solutions, but in this case it will simply be a check of our work and not a search for extraneous solutions.

Example 6

Solving a Radical Equation

NOTE Because a *cube root* is involved, we *cube* both sides to remove the radical.

Solve $\sqrt[3]{x^2 + 23} = 3.$

Cubing each side, we have

$$x^2 + 23 = 27$$

which results in the quadratic equation

$$x^2 - 4 = 0$$

This has two solutions

$$x = 2 \qquad \text{or} \qquad x = -2$$

Checking the solutions, we find that both result in true statements. Again you should verify this result.

CHECK YOURSELF 6

Solve $\sqrt[3]{x^2 - 8} - 2 = 0.$

We summarize our work in this section in the following algorithm for solving equations involving radicals.

Step by Step: Solving Equations Involving Radicals

Step 1 Isolate a radical on one side of the equation.
Step 2 Raise each side of the equation to the smallest power that will eliminate the isolated radical.
Step 3 If any radicals remain in the equation derived in step 2, return to step 1 and continue the solution process.
Step 4 Solve the resulting equation to determine any possible solutions.
Step 5 Check all solutions to determine whether extraneous solutions may have resulted from step 2.

Did you ever stand on a beach and wonder how far out into the ocean you could see? Or have you wondered how close a ship has to be to spot land? In either case, the function

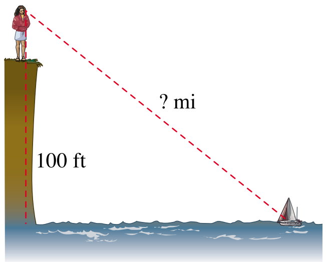
$$d(h) = \sqrt{2h}$$

can be used to estimate the distance to the horizon (in miles) from a given height (in feet).

Example 7

Estimating a Distance

Cordelia stood on a cliff gazing out at the ocean. Her eyes were 100 ft above the ocean. She saw a ship on the horizon. Approximately how far was she from that ship?



Substituting 100 for h in the equation, we get

$$d(h) = \sqrt{2(100)}$$

$$d(h) = \sqrt{200}$$

$$d(h) \approx 14 \text{ mi}$$

CHECK YOURSELF 7

From a plane flying at 35,000 ft, how far away is the horizon?

CHECK YOURSELF ANSWERS

1. $\{21\}$ **2.** No solution **3.** $\{9\}$ **4.** $\left\{0, -\frac{1}{9}\right\}$ **5.** $\{6\}$ **6.** $\{4, -4\}$

7. $d(h) \approx 265$ mi

Name ____________

Section ________ Date ________

8.4 Exercises

Solve each of the following equations. Be sure to check your solutions.

1. $\sqrt{x} = 2$

2. $\sqrt{x} - 3 = 0$

3. $2\sqrt{y} - 1 = 0$

4. $3\sqrt{2z} = 9$

5. $\sqrt{m + 5} = 3$

6. $\sqrt{y + 7} = 5$

7. $\sqrt{2x + 4} - 4 = 0$

8. $\sqrt{3x + 3} - 6 = 0$

9. $\sqrt{3x - 2} + 2 = 0$

10. $\sqrt{4x + 1} + 3 = 0$

11. $\sqrt{x - 1} = \sqrt{1 - x}$

12. $\sqrt{x + 1} = \sqrt{1 + x}$

13. $\sqrt{w + 3} = \sqrt{3 + w}$

14. $\sqrt{w - 3} = \sqrt{3 - w}$

15. $\sqrt{2x - 3} + 1 = 3$

16. $\sqrt{3x + 1} - 2 = -1$

17. $2\sqrt{3z + 2} - 1 = 5$

18. $3\sqrt{4q - 1} - 2 = 7$

19. $\sqrt{15 - 2x} = x$

20. $\sqrt{48 - 2y} = y$

21. $\sqrt{x + 5} = x - 1$

22. $\sqrt{2x - 1} = x - 8$

ANSWERS

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________
8. ____________
9. ____________
10. ____________
11. ____________
12. ____________
13. ____________
14. ____________
15. ____________
16. ____________
17. ____________
18. ____________
19. ____________
20. ____________
21. ____________
22. ____________

ANSWERS

23. ______
24. ______
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36. ______
37. ______
38. ______
39. ______
40. ______
41. ______
42. ______
43. ______
44. ______

23. $\sqrt{3m - 2} + m = 10$

24. $\sqrt{2x + 1} + x = 7$

25. $\sqrt{t + 9} + 3 = t$

26. $\sqrt{2y + 7} + 4 = y$

27. $\sqrt{6x + 1} - 1 = 2x$

28. $\sqrt{7x + 1} - 1 = 3x$

29. $\sqrt[3]{x - 5} = 3$

30. $\sqrt[3]{x + 6} = 2$

31. $\sqrt[3]{x^2 - 1} = 2$

32. $\sqrt[3]{x^2 + 11} = 3$

Solve each of the following equations. Be sure to check your solutions.

33. $\sqrt{2x} = \sqrt{x + 1}$

34. $\sqrt{3x} = \sqrt{5x - 1}$

35. $2\sqrt{3r} = \sqrt{r + 11}$

36. $5\sqrt{2q - 7} = \sqrt{15q}$

37. $\sqrt{x + 2} + 1 = \sqrt{x + 4}$

38. $\sqrt{x + 5} - 1 = \sqrt{x + 3}$

39. $\sqrt{4m - 3} - 2 = \sqrt{2m - 5}$

40. $\sqrt{2c - 1} = \sqrt{3c + 1} - 1$

41. $\sqrt{x + 1} + \sqrt{x} = 1$

42. $\sqrt{z - 1} - \sqrt{6 - z} = 1$

43. $\sqrt{5x + 6} - \sqrt{x + 3} = 3$

44. $\sqrt{5y + 6} - \sqrt{3y + 4} = 2$

45. $\sqrt{y^2 + 12y} - 3\sqrt{5} = 0$

46. $\sqrt{x^2 + 2x} - 2\sqrt{6} = 0$

47. $\sqrt{\dfrac{x - 3}{x + 2}} = \dfrac{2}{3}$

48. $\dfrac{\sqrt{x - 2}}{x - 2} = \dfrac{x - 5}{\sqrt{x - 2}}$

49. $\sqrt{\sqrt{t} + 5} = 3$

50. $\sqrt{\sqrt{s} - 1} = \sqrt{s - 7}$

51. For what values of x is $\sqrt{(x - 1)^2} = x - 1$ a true statement?

52. For what values of x is $\sqrt[3]{(x - 1)^3} = x - 1$ a true statement?

Solve for the indicated variable.

53. $h = \sqrt{pq}$ for q

54. $c = \sqrt{a^2 + b^2}$ for a

55. $v = \sqrt{2gR}$ for R

56. $v = \sqrt{2gR}$ for g

57. $r = \sqrt{\dfrac{S}{2\pi}}$ for S

58. $r = \sqrt{\dfrac{3V}{4\pi}}$ for V

59. $r = \sqrt{\dfrac{2V}{\pi h}}$ for V

60. $r = \sqrt{\dfrac{2V}{\pi h}}$ for h

61. $d = \sqrt{(x - 1)^2 + (y - 2)^2}$ for x

62. $d = \sqrt{(x - 1)^2 + (y - 2)^2}$ for y

ANSWERS

45. ______
46. ______
47. ______
48. ______
49. ______
50. ______
51. ______
52. ______
53. ______
54. ______
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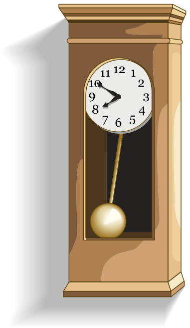
56. ______
57. ______
58. ______
59. ______
60. ______
61. ______
62. ______

ANSWERS

63. ______
64. ______
65. ______
66. ______
67. ______
68. ______
69. ______
70. ______

A weight suspended on the end of a string is a *pendulum*. The most common example of a pendulum (this side of Edgar Allen Poe) is the kind found in many clocks.

The regular back-and-forth motion of the pendulum is *periodic,* and one such cycle of motion is called a *period*. The time, in seconds, that it takes for one period is given by the radical equation



$$t = 2\pi\sqrt{\frac{l}{g}}$$

in which g is the force of gravity (10 m/s^2) and l is the length of the pendulum.

63. Find the period (to the nearest hundredth of a second) if the pendulum is 0.9 m long.

64. Find the period if the pendulum is 0.049 m long.

65. Solve the equation for length l.

66. How long would the pendulum be if the period were exactly 1 s?

Solve each of the following applications.

67. The sum of an integer and its square root is 12. Find the integer.

68. The difference between an integer and its square root is 12. What is the integer?

69. The sum of an integer and twice its square root is 24. What is the integer?

70. The sum of an integer and 3 times its square root is 40. Find the integer.

71. If a plane flies at 30,000 ft, how far away is the horizon?

72. Janine was looking out across the ocean from her hotel room on the beach. Her eyes were 250 ft above the ground. She saw a ship on the horizon. Approximately how far was the ship from her?

73. Given a distance, d, to the horizon, what altitude would allow you to see that far?

When a car comes to a sudden stop, you can determine the skidding distance (in feet) for a given speed (in miles per hour) using the formula $s(x) = 2\sqrt{5x}$, in which s is skidding distance and x is speed. Calculate the skidding distance for the following speeds.

74. 55 mi/h

75. 65 mi/h

76. 75 mi/h

77. 40 mi/h

78. Given the skidding distance s, what formula would allow you to calculate the speed in miles per hour?

79. Use the formula obtained in exercise 78 to determine the speed of a car in miles per hour if the skid marks were 35 ft long.

For each given equation, use a graphing calculator to solve. Express solutions to the nearest hundredth. (*Hint:* Define Y_1 by the expression on the left side of the equation and define Y_2 by the expression on the right side. Graph these functions and locate any intersection points. For each such point, the x value represents a solution.)

80. $\sqrt{x + 4} = x - 3$

81. $\sqrt{2 - x} = x + 4$

82. $3 - 2\sqrt{x + 4} = 2x - 5$

83. $5 - 3\sqrt{2 - x} = 3 - 4x$

ANSWERS

71. ______
72. ______
73. ______
74. ______
75. ______
76. ______
77. ______
78. ______
79. ______
80. ______
81. ______
82. ______
83. ______

Answers

1. 4 **3.** $\frac{1}{4}$ **5.** 4 **7.** 6 **9.** No solution **11.** 1

13. All real numbers **15.** $\frac{7}{2}$ **17.** $\frac{7}{3}$ **19.** 3 **21.** 4 **23.** 6

25. 7 **27.** $0, \frac{1}{2}$ **29.** 32 **31.** ± 3 **33.** 1 **35.** 1 **37.** $-\frac{7}{4}$

39. 3, 7 **41.** 0 **43.** 6 **45.** $-15, 3$ **47.** 7 **49.** 16 **51.** $x \geq 1$

53. $q = \frac{h^2}{p}$ **55.** $R = \frac{v^2}{2g}$ **57.** $S = 2\pi r^2$ **59.** $V = \frac{\pi h r^2}{2}$

61. $x = 1 \pm \sqrt{d^2 - (y - 2)^2}$ **63.** 1.88 s **65.** $l = \frac{t^2 g}{4\pi^2}$ **67.** 9

69. 16 **71.** $\approx$245 mi **73.** $\frac{d^2}{2}$ **75.** $\approx$36 ft **77.** $\approx$28 ft

79. $\approx$60 mi/h **81.** $\{-2\}$ **83.** $\{0.44\}$