The applications found here will mostly involve exponential equations. These equations will be solved using logarithms and their properties.

Interest Problems

Compound Interest – If we start with a principal of P dollars then the amount A in an account after t years, with an annual interest rate r compounded n times a year, is given by:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A. Use the compound interest formula to solve the following.

If a \$500 certificate of deposit earns $4\frac{1}{4}\%$ compounded monthly then how much will be accumulated at the end of a 3 year period?

Given $P = \$500$

$r = 4\frac{1}{4}\% = 4.25\% = .0425$

$n = 12$

$t = 3\text{yrs}$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= 500\left(1+\frac{.0425}{12}\right)^{12\cdot 3}$$

$$= 500\left(1+\frac{.0425}{12}\right)^{36}$$

$$\approx 567.86$$

At the end of 3 years the Amount is \$567.86.

A certain investment earns $8\frac{3}{4}\%$ compounded quarterly. If \$10,000 dollars is invested for 5 years, how much will be in the account at the end of that time period?

Given $P = \$10{,}000$

$r = 8\frac{3}{4}\% = 8.75\% = .0875$

$n = 4$

$t = 5\text{yrs}$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= 10{,}000\left(1+\frac{.0875}{4}\right)^{4\cdot 5}$$

$$= 10{,}000\left(1+\frac{.0875}{4}\right)^{20}$$

$$\approx 15{,}415.42$$

At the end of 5 years the Account will be worth \$15,415.42

The basic idea is to first determine the given information then substitute the appropriate values into the formula and evaluate. To avoid round-off error, use the use the calculator and round-off ***only once*** as the last step.

Compounded Annually	$n = 1$	Compounded Semiannually	$n = 2$
Compounded Quarterly	$n = 4$	Compounded Monthly	$n = 12$
Compounded Daily	$n = 365$		

One important application is to determine the ***doubling time***. How long does it take for the principal, in a compound interest account, to double?

How long does it take to double $1000 at an annual interest rate of 6.35% compounded monthly?

Given $P = \$1,000$

$r = 6.35\% = .0635$

$n = 12$

$A = \$2,000$ (Double)

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$2000 = 1000\left(1 + \frac{.0635}{12}\right)^{12t}$$

$$2 = \left(1 + \frac{.0635}{12}\right)^{12t}$$

$$\log 2 = \log\left(1 + \frac{.0635}{12}\right)^{12t}$$

$$\log 2 = 12t \cdot \log\left(1 + \frac{.0635}{12}\right)$$

$$\frac{\log 2}{12 \log\left(1 + \frac{.0635}{12}\right)} = t$$

$$t \approx 10.9 \text{ years}$$

The account will Double in about 10.9 years.

Take the common log of both sides and apply the power rule.

The key step in this process is to take the common log of both sides so that we can apply the power rule and solve for *t*. Only use the calculator in the last step and round-off only once.

How long will it take $30,000 to accumulate to $110,000 in a trust that earns a 10% annual return compounded semiannually?

Given $P = \$30,000$

$r = 10\% = .10$

$n = 2$

$A = \$110,000$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$110,000 = 30,000\left(1 + \frac{.10}{2}\right)^{2t}$$

$$\frac{110,000}{30,000} = \left(1 + \frac{.10}{2}\right)^{2t}$$

$$\log\left(\frac{11}{3}\right) = 2t \cdot \log(1 + .05)$$

$$\frac{\log\left(\frac{11}{3}\right)}{2 \cdot \log(1.05)} = t$$

$$t \approx 13.3$$

Ans: It will take about 13.3 years

How long will it take our money to triple in a bank account with an annual interest rate of 8.45% compounded annually?

Given $r = 8.45\% = .0845$

and $n = 1$

Assume $A = 3P$ (triple)

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$3P = P\left(1 + \frac{.0845}{1}\right)^{1 \cdot t}$$

$$3 = (1.0845)^t$$

$$\log 3 = t \cdot \log(1.0845)$$

$$\frac{\log 3}{\log(1.0845)} = t$$

$$t \approx 13.5$$

Ans: About 13.5 years to triple.

Make a note that doubling or tripling time is independent of the principal. In the previous problem, notice that the principal was not given and also notice that the *P* cancelled.

Continuously Compounding Interest – If we start with a principal of *P* dollars then the amount *A* in an account after *t* years, with an annual interest rate *r* compounded continuously, is given by: $A = Pe^{rt}$

B. Use the continuously compounding interest formula to solve the following.

If a \$500 certificate of deposit earns $4\frac{1}{4}\%$ compounded continuously then how much will be accumulated at the end of a 3 year period?

Given $P = \$500$

$r = 4\frac{1}{4}\% = 4.25\% = .0425$

$t = 3$

$$A = Pe^{rt}$$
$$= 500e^{.0425(3)}$$
$$= 500e^{.1275}$$
$$\approx 567.99$$

The amount at the end of 3yrs will be \$567.99

A certain investment earns $8\frac{3}{4}\%$ compounded continuously. If \$10,000 dollars is invested for 5 years how much will be in the account after 5 years?

Given $P = \$10,000$

$r = 8\frac{3}{4}\% = 8.75\% = .0875$

$t = 5$

$$A = Pe^{rt}$$
$$= 10,000e^{.0875(5)}$$
$$= 10,000e^{.4375}$$
$$\approx 15,488.30$$

The amount at the end of 5yrs will be \$15,488.30

The previous two examples are the same examples that we started this chapter with. This allows us to compare the accumulated amounts to that of regular compound interest.

Continuous Compounding	n Compoundings per year		Difference
\$567.99	\$567.86	(n = 12)	\$0.13
\$15,488.30	\$15,415.42	(n = 4)	\$72.88

As we can see, continuous compounding is better, but not by much. Instead of buying a new car for say \$20,000, let us invest in the future of our family. If we invest the \$20,000 at 6% annual interest compounded continuously for say, two generations or 100 years, then how much will our family have accumulated in that time?

$$A = Pe^{rt} = 20,000e^{.06(100)} = 20,000e^{6} = 8,068,575.87$$

The answer is over 8 million dollars. One can only wonder actually how much that would be worth in a century.

Given continuous compounding we could also determine the ***doubling time***. Instead of taking the common log of both sides it will be easier take the natural log of both sides, otherwise the steps are the same.

How long does it take to double \$1000 at an annual interest rate of 6.35% compounded continuously?
Given $P = \$1,000$ $r = 6.35\% = .0635$ $A = \$2,000$ Ans: The Account will double in about 10.9yrs. $A = Pe^{rt}$ $2000 = 1000e^{.0635 \cdot t}$ $2 = e^{.0635t}$ $\ln 2 = \ln e^{.0635t}$ $\ln 2 = .0635t \cdot \ln e$ $\ln 2 = .0635t \cdot 1$ $\ln 2/.0635 = t$ $t \approx 10.9$

Take the natural log of both sides and apply the power rule.

$\ln e = 1$

The key step in this process is to take the natural log of both sides so that we can apply the power rule and solve for *t*. Only use the calculator in the last step and round-off only once.

How long will it take \$30,000 to accumulate to \$110,000 in a trust that earns a 10% annual return compounded continuously?
Given $P = \$30,000$ $r = 10\% = .10$ $A = \$110,000$ Ans: Approximately 13 years. $A = Pe^{rt}$ $110,000 = 30,000e^{.10 \cdot t}$ $\frac{110,000}{30,000} = e^{.10t}$ $\ln\left(\frac{11}{3}\right) = .10t \ln e$ $\frac{\ln(11/3)}{.10} = t \cdot 1$ $t \approx 13$ yrs

How long will it take our money to triple in a bank account with an annual interest rate of 8.45% compounded continuously?
Given $r = 8.45\% = .0845$ and $A = 3P$ Ans: Approximately 13 years. $A = Pe^{rt}$ $3P = Pe^{.0845t}$ $3 = e^{.0845t}$ $\ln 3 = .0845t \cdot \ln e$ $\frac{\ln 3}{.0845} = t$ $t \approx 13$

Exponential Growth/Decay Problems

Continuously compounding interest is an example of exponential growth. This idea can be extended to a multitude of applications.

Exponential Growth / Decay Model – Given exponential growth or decay the amount P after time t is given by the following formula: $P = P_0 e^{kt}$

Here P_0 is the initial amount and k is the exponential growth/decay rate.

If k is positive then we will have a growth model and if k is negative then we will have a decay model.

C. Use the exponential growth/decay model to answer the questions.

A certain bacterium has an exponential growth rate of 25% per day. If we start with 0.5 gram and provide unlimited resources how much bacteria can we grow in 2 weeks?

Given $P_0 = .5\text{gm}$

$k = 25\% = .25$

$t = 14$ days

$P = P_0 e^{kt}$

$= .5e^{.25(14)}$

$= .5e^{3.5} \approx 16.56$

We can grow 16.56 gms in 2 weeks.

Typically the exponential growth rate will not be given. In this case we must determine that before we can use the model to answer the question.

During its exponential growth phase, a certain bacterium can grow from 5,000 cells to 12,000 cells in 10 hours. At this rate how many cells will be present after 36 hours?

Given $P_0 = 5{,}000$ cells

① find k: (12,000 cells in 10hrs)

$P = P_0 e^{kt}$

$12{,}000 = 5000e^{k\cdot 10}$

$\frac{12{,}000}{5{,}000} = e^{10k}$

$\ln\left(\frac{12}{5}\right) = 10k \ln e$

$\frac{\ln\left(\frac{12}{5}\right)}{10} = k$

$k \approx .087547$

***Step 1**: Use the given information to calculate the growth/ decay rate k.*

$\ln e = 1$

② Model:

$P = 5{,}000e^{\left(\frac{\ln(2.4)}{10}\right)\cdot t}$

***Step 2**: Substitute the initial amount and k to formulate a model.*

③ Answer the question.

How many in 36 hours?

$P = 5{,}000e^{\left(\frac{\ln 2.4}{10}\right)(36)}$

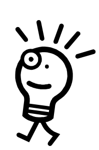
$= 5{,}000e^{3.6\ln 2.4}$

$\approx 116{,}877$

Ans: About 116,877 cells in 36 hours

***Step 3**: Use the model to answer the question.*

Tip: Use the exact value for k and avoid round-off error. If we use the approximate, rounded-off, value for k we will compound the error by rounding off again at the end when calculating the final result.



During its exponential growth phase, a certain bacterium can grow from 5,000 cells to 12,000 cells in 10 hours. At this rate how long will it take to grow to 50,000 cells?

Given $P_0 = 5{,}000$

① find K (12,000 cells in 10 hrs)

$P = P_0 e^{kt}$

$12{,}000 = 5000\, e^{k\cdot 10}$

$\frac{12{,}000}{5{,}000} = e^{10k}$

$\ln(12/5) = 10k \cdot \ln e$

$K = \frac{\ln(2.4)}{10}$

② Model:

$P = 5{,}000 e^{\left(\frac{\ln(2.4)}{10} \cdot t\right)}$

③ Find t when $P = 50{,}000$ cells

$50{,}000 = 5{,}000\, e^{\frac{\ln 2.4}{10} \cdot t}$

$\frac{50{,}000}{5{,}000} = e^{\frac{\ln 2.4}{10} \cdot t}$

$\ln(10) = \frac{\ln 2.4}{10} \cdot t \cdot \cancel{\ln e}^{1}$

$\frac{10 \cdot \ln(10)}{\ln(2.4)} = t$

$t \approx 26.3$

Ans: We will have 50,000 cells in approximately 26.3 hours.

The population of a certain city in 1975 was 65,000. In 2000 the census determined that the population was 99,500. Assuming exponential growth, estimate the population in 2015.

Given $P_0 = 65{,}000$ people

① find K: 99,500 people in 25 yrs

$P = P_0 e^{kt}$

$99{,}500 = 65{,}000 e^{k\cdot 25}$

$\frac{99{,}500}{65{,}000} = e^{25K}$

$\ln\left(\frac{995}{650}\right) = 25K \ln e$

$K = \ln\left(\frac{995}{650}\right)/25$

② model

$P = 65{,}000 e^{\left(\ln\left(\frac{995}{650}\right)/25 \cdot t\right)}$

③ Find P when $t = 40$ yrs

$P = 65{,}000 e^{\left(\ln\left(\frac{995}{650}\right)/25 \cdot (40)\right)}$

$= 65{,}000^{1.6 \ln\left(\frac{995}{650}\right)}$

$\approx 128{,}460$

Estimate 128,460 people in 2015.

A certain animal species can double its population every 30 years. Assuming exponential growth, how long will it take the population to grow from 40 specimens to 500?

① find K: $P = P_0 e^{kt}$

$2P_0 = P_0 e^{k\cdot 30}$

$2 = e^{30k}$

$\ln 2 = 30k \ln e$

$\ln 2/30 = K$

② model: $P = 40 e^{\left(\frac{\ln 2}{30} \cdot t\right)}$

③ Find t when $P = 500$

$500 = 40\, e^{\left(\frac{\ln 2}{30} \cdot t\right)}$

$\frac{500}{40} = e^{\frac{\ln 2}{30} \cdot t}$

$\ln\left(\frac{500}{40}\right) = \frac{\ln 2}{30} \cdot t \cdot \cancel{\ln e}^{1}$

$\frac{30 \ln(12.5)}{\ln 2} = t$

$t \approx 109.3$

Ans: 500 specimens in about 109.3 years.

Up to this point we have seen only exponential growth. We will conclude this section with some exponential decay applications. Often exponential rate of decay can be gotten from the half-life information. ***Half-life*** is the amount of time it takes for a substance to decay to half of the original amount.

If certain isotope has a half-life of 4.2 days. How long will it take for a 150 milligram sample to decay so that only 10 milligrams are left?

Given $P_0 = 150$ mg

① find K given half-life of 4.2 days

$P = P_0 e^{kt}$

$75 = 150 e^{k(4.2)}$

$\frac{1}{2} = e^{4.2k}$

$\ln\left(\frac{1}{2}\right) = 4.2k \ln e$

$\frac{\ln\left(\frac{1}{2}\right)}{4.2} = k$

$k \approx -.165035$

② model: $P = 150 e^{\left(\frac{\ln(.5)}{4.2} \cdot t\right)}$

③ Find t when $P = 10$ mg

$10 = 150 e^{\left(\frac{\ln(.5)}{4.2} \cdot t\right)}$

$\frac{10}{150} = e^{\frac{\ln(.5)}{4.2} \cdot t}$

$\ln\left(\frac{1}{15}\right) = \frac{\ln(.5)}{4.2} t \cdot \cancel{\ln e}^{1}$

$\frac{4.2 \ln\left(\frac{1}{15}\right)}{\ln(.5)} = t$

$t \approx 16.4$

It will take about 16.4 days.

The half-life of carbon-14 is 5730 years. If it is determined that an old bone contains 85% of it original carbon-14 how old is the bone?

Given $P = .85 P_0$

① find k given half-life of $C^{14} \rightarrow 5730$ yrs

$P = P_0 e^{kt}$

$\frac{1}{2} P_0 = P_0 e^{k \cdot 5730}$

$\frac{1}{2} = e^{k \cdot 5730}$

$\ln\left(\frac{1}{2}\right) = k \cdot 5730 \cancel{\ln e}^{1}$

$\frac{\ln(.5)}{5730} = k$

② model: $P = P_0 e^{\left(\frac{\ln(.5)}{5730} \cdot t\right)}$

③ Find t when $P = .85 P_0$

$.85 P_0 = P_0 e^{\frac{\ln(.5)}{5730} \cdot t}$

$.85 = e^{\frac{\ln(.5)}{5730} \cdot t}$

$\ln(.85) = \frac{\ln(.5)}{5730} \cdot t \ln e$

$\frac{5730 \ln(.85)}{\ln(.5)} = t$

$t \approx 1{,}343.5$

Ans: It takes about 1,343.5 yrs for a bone to lose 15% of its carbon-14.

To summarize, first find the growth/decay rate. Put together a mathematical model using the initial amount and the exponential rate of growth/decay. Then use the model to answer the question.

Acidity Model – $pH = -\log(H^+)$

PH is a measure of the hydrogen ion concentration H^+ in moles of hydrogen per liter.

Remember that a logarithm without an indicated base is assumed to be base 10, the common logarithm.

D. Use the acidity model to answer the questions.

Find the pH of a solution that has a hydrogen ion concentration of 8.7×10^{-8} moles/liter.

Given $H^+ = 8.7 \times 10^{-8}$

$$pH = -\log(H^+)$$
$$= -\log(8.7 \times 10^{-8}) \approx 7.0605$$

Here we will have to solve logarithmic equations so the process is a bit different. After we isolate the logarithm we will then apply the definition. This will leave us with an exponential equation.

A certain fruit has a pH of 2.2 and an antacid tablet has a pH of 10.1. How many times more is the concentration of hydrogen ion in fruit as compared to the antacid?

<u>Fruit pH=2.2</u>

$$pH = -\log(H_f^+)$$
$$2.2 = -\log(H_f^+)$$
$$-2.2 = \log(H_f^+)$$
$$10^{-2.2} = H_f^+ \quad \begin{cases}\text{Larger} \\ \text{concentration}\end{cases}$$

<u>Antacid pH=10.1</u>

$$pH = -\log(H_a^+)$$
$$10.1 = -\log(H_a^+)$$
$$-10.1 = \log(H_a^+)$$
$$10^{-10.1} = H_a^+$$

Apply the definition of the logarithm here.

$$x \cdot H_a^+ = H_f^+$$
$$x = \frac{H_f^+}{H_a^+} = \frac{10^{-2.2}}{10^{-10.1}} = 10^{7.9}$$

Fruit has a hydrogen ion concentration that is $10^{7.9}$ more than an antacid tablet.

An alternative method for solving the logarithmic equation above is to use the property that says if $x = y$ then $10^x = 10^y$.

$$-10.1 = \log(H_a^+)$$
$$10^{-10.1} = 10^{\log(H_a^+)}$$
$$10^{-10.1} = H_a^+$$

Volume of Sound Model – $L = 10 \cdot \log\left(\frac{I}{10^{-12}}\right)$

Here the volume L is measured in decibels (db) and I is the intensity in watts per square meter (W/m^2).

E. Use the volume of sound model to answer the questions.

An alarm has an intensity of 5.8×10^{-9} W/m^2. How loud is the alarm in decibels?

$$L = 10 \cdot \log\left(\frac{I}{10^{-12}}\right)$$
$$= 10 \cdot \log\left(\frac{5.8 \times 10^{-9}}{10^{-12}}\right)$$
$$= 10 \cdot \log(5.8 \times 10^{3}) \approx 37.6 \text{ db}$$

Anna can scream at 56 db and Billy can yell at 48 db. How many more times intense is Anna's scream than Billy's yell?

Anna's $L = 56$ db

$$L = 10 \cdot \log\left(\frac{I_a}{10^{-12}}\right)$$
$$56 = 10 \cdot \log\left(\frac{I_a}{10^{-12}}\right)$$
$$5.6 = \log\left(\frac{I_a}{10^{-12}}\right)$$
$$10^{5.6} = \frac{I_a}{10^{-12}}$$
$$10^{5.6} \cdot 10^{-12} = I_a$$
$$10^{-6.4} = I_a \quad \{\text{Larger Intensity}$$

Billy's $L = 48$ db

$$L = 10 \cdot \log\left(\frac{I_b}{10^{-12}}\right)$$
$$48 = 10 \cdot \log\left(\frac{I_b}{10^{-12}}\right)$$
$$4.8 = \log\left(\frac{I_b}{10^{-12}}\right)$$
$$10^{4.8} = \frac{I_b}{10^{-12}}$$
$$10^{4.8} \cdot 10^{-12} = I_b$$
$$10^{-7.2} = I_b$$

$$x \cdot I_b = I_a$$
$$x = \frac{I_a}{I_b} = \frac{10^{-6.4}}{10^{-7.2}} = 10^{.8} \approx 6.3$$

Anna's scream is 6.3 times as intense as billy's.