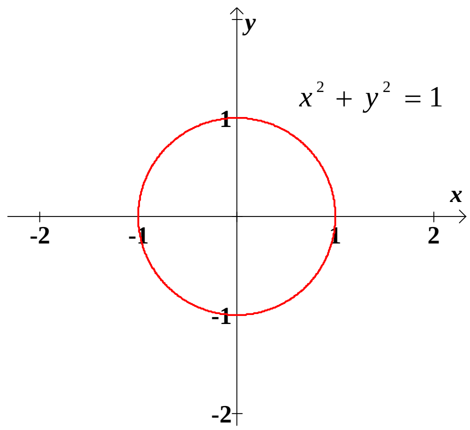
UNIT CIRCLE TRIGONOMETRY

The Unit Circle is the circle centered at the origin with radius 1 unit (hence, the "unit" circle). The equation of this circle is $x^2 + y^2 = 1$. A diagram of the unit circle is shown below:

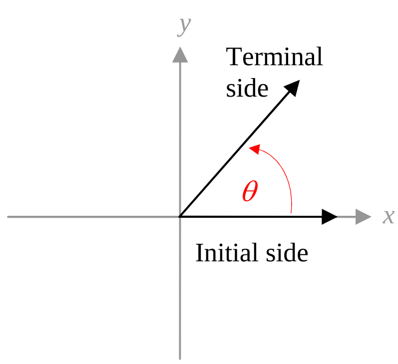


We have previously applied trigonometry to triangles that were drawn with no reference to any coordinate system. Because the radius of the unit circle is 1, we will see that it provides a convenient framework within which we can apply trigonometry to the coordinate plane.

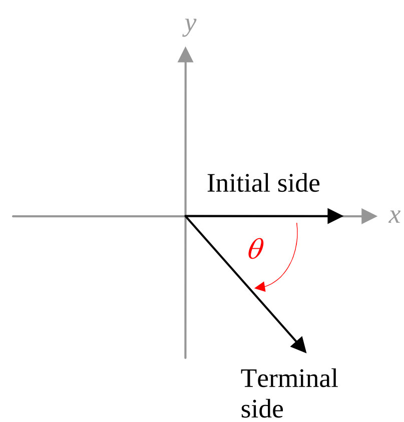
Drawing Angles in Standard Position

We will first learn how angles are drawn within the coordinate plane. An angle is said to be in <u>standard position</u> if the vertex of the angle is at (0, 0) and the initial side of the angle lies along the positive *x*-axis. If the angle measure is positive, then the angle has been created by a counterclockwise rotation from the initial to the terminal side. If the angle measure is negative, then the angle has been created by a clockwise rotation from the initial to the terminal side.

θ in standard position, where θ is positive:



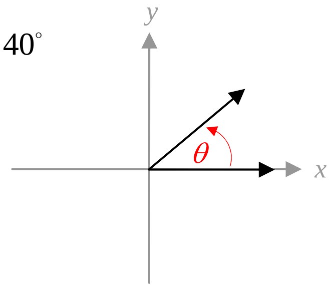
θ in standard position, where θ is negative:



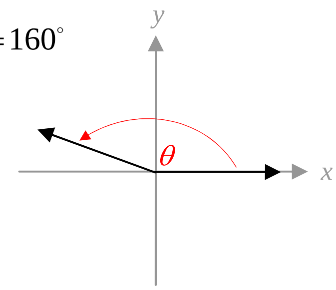
Examples

The following angles are drawn in standard position:

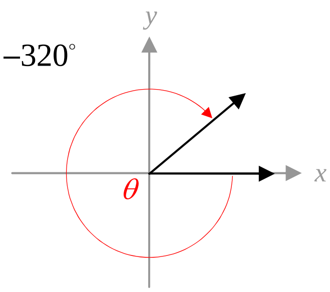
1. $\theta = 40^\circ$



2. $\theta = 160^\circ$



3. $\theta = -320^\circ$

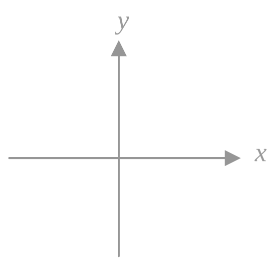


Notice that the terminal sides in examples 1 and 3 are in the same position, but they do not represent the same angle (because the amount and direction of the rotation in each is different). Such angles are said to be *<u>coterminal.</u>*

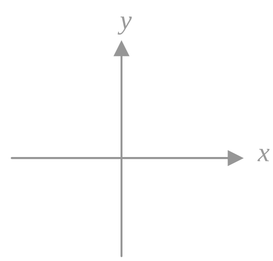
Exercises

Sketch each of the following angles in standard position. (Do not use a protractor; just draw a brief sketch.)

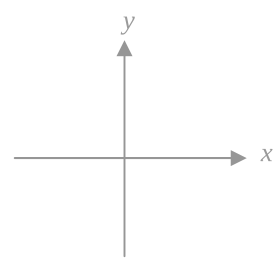
1. $\theta = 120^\circ$



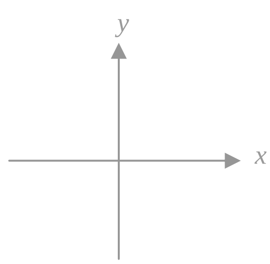
2. $\theta = -45^\circ$



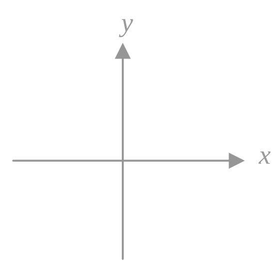
3. $\theta = -130^\circ$



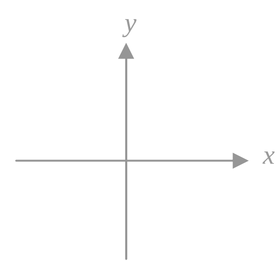
4. $\theta = 270^\circ$



$\theta = -90^\circ$



6. $\theta = 750^\circ$

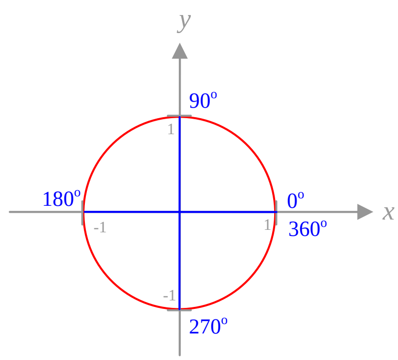


Labeling Special Angles on the Unit Circle

We are going to deal primarily with special angles around the unit circle, namely the multiples of 30°, 45°, 60°, and 90°. All angles throughout this unit will be drawn in standard position.

First, we will draw a unit circle and label the angles that are multiples of 90°. These angles, known as <u>quadrantal angles</u>, have their terminal side on either the *x*-axis or the *y*-axis. (We have limited our diagram to the quadrantal angles from 0° to 360°.)

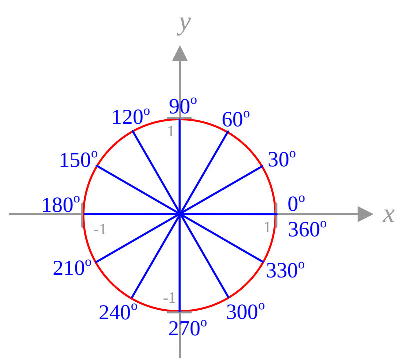
<u>Multiples of 90°:</u>



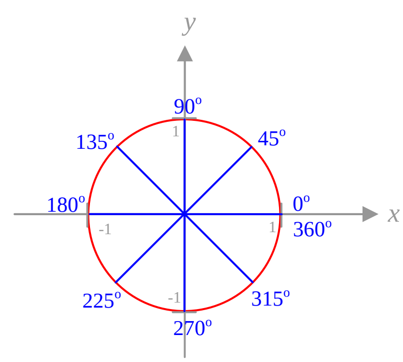
Note: The 0° angle is said to be <u>coterminal</u> with the 360° angle. (Coterminal angles are angles drawn in standard position that share a terminal side.)

Next, we will repeat the same process for multiples of 30°, 45°, and 60°. (Notice that there is a great deal of overlap between the diagrams.)

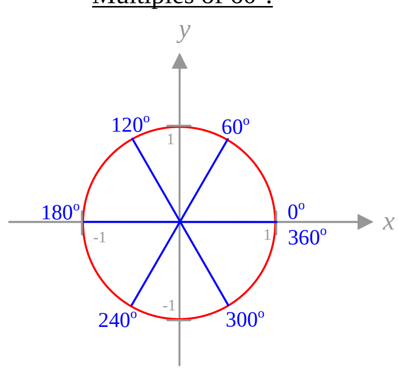
<u>Multiples of 30°:</u>



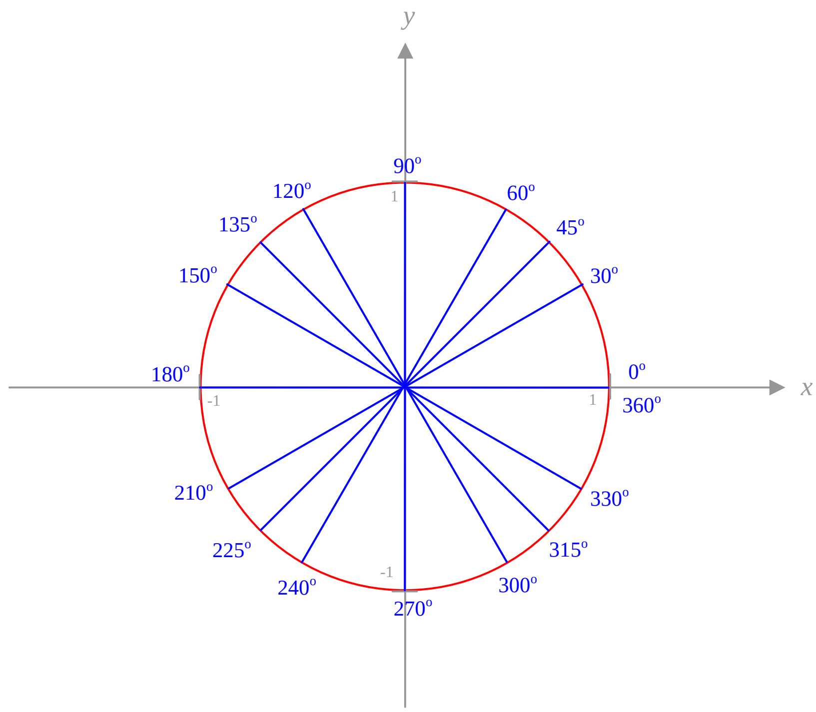
<u>Multiples of 45°:</u>



<u>Multiples of 60°:</u>

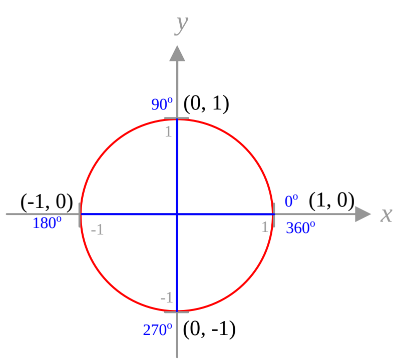


Putting it all together, we obtain the following unit circle with all special angles labeled:



Coordinates of Quadrantal Angles and First Quadrant Angles

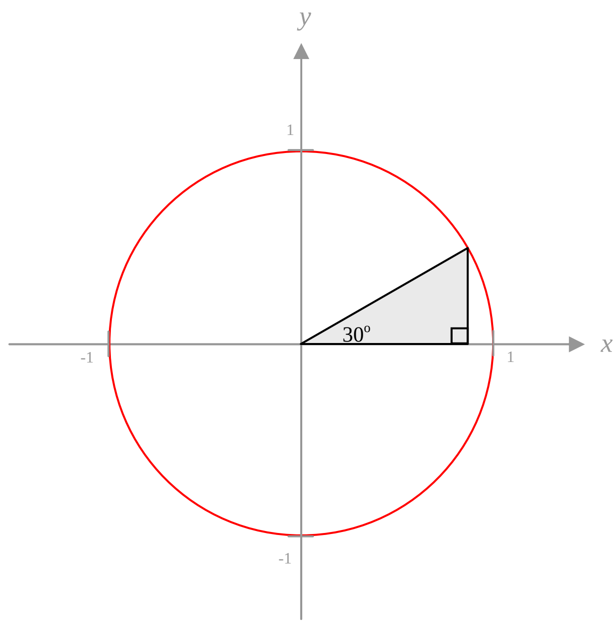
We want to find the coordinates of the points where the terminal side of each of the quadrantal angles intersects the unit circle. Since the unit circle has radius 1, these coordinates are easy to identify; they are listed in the table below.



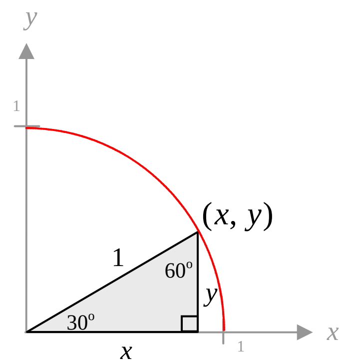
Angle	Coordinates
0^o	(1, 0)
90^o	(0, 1)
180^o	(-1, 0)
270^o	(0, -1)
360^o	(1, 0)

We will now look at the first quadrant and find the coordinates where the terminal side of the 30^o, 45^o, and 60^o angles intersects the unit circle.

First, we will draw a right triangle that is based on a 30^o reference angle. (When an angle is drawn in standard position, its <u>reference angle</u> is the positive acute angle measured from the *x*-axis to the angle's terminal side. The concept of a reference angle is crucial when working with angles in other quadrants and will be discussed in detail later in this unit.)



Notice that the above triangle is a 30^o-60^o -90^o triangle. Since the radius of the unit circle is 1, the hypotenuse of the triangle has length 1. Let us call the horizontal side of the triangle *x*, and the vertical side of the triangle *y*, as shown below. (Only the first quadrant is shown, since the triangle is located in the first quadrant.)

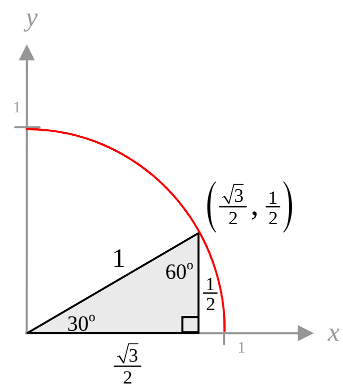


We want to find the values of *x* and *y*, so that we can ultimately find the coordinates of the point (x, y) where the terminal side of the 30^o angle intersects the unit circle. Recall our theorem about 30^o-60^o-90^o triangles:

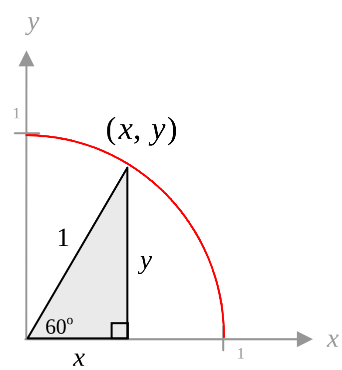
In a 30^o-60^o-90^o triangle, the length of the hypotenuse is twice the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.

Since the length of the hypotenuse is 1 and it is twice the length of the shorter leg, *y*, we can say that $y = \frac{1}{2}$. Since the longer leg, *x*, is $\sqrt{3}$ times the length of the shorter leg, we can say that $x = \frac{1}{2}\sqrt{3}$, or equivalently, $x = \frac{\sqrt{3}}{2}$.

Based on the values of the sides of the triangle, we now know the coordinates of the point (x, y) where the terminal side of the 30^o angle intersects the unit circle. This is the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, as shown below.

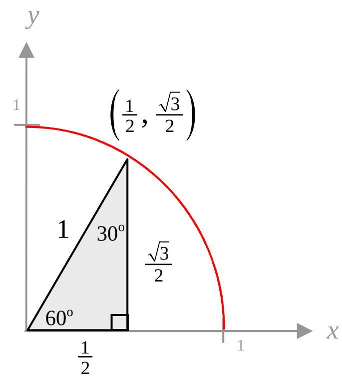


We will now repeat this process for a 60^o reference angle. We first draw a right triangle that is based on a 60^o reference angle, as shown below.

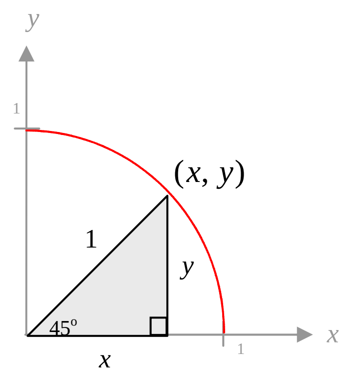


We again want to find the values of *x* and *y*. The triangle is a 30^o-60^o -90^o triangle. Since the length of the hypotenuse is 1 and it is twice the length of the shorter leg, *x*, we can say that $x = \frac{1}{2}$. Since the longer leg, *y*, is $\sqrt{3}$ times the length of the shorter leg, we can say that $y = \frac{1}{2}\sqrt{3}$, or equivalently, $y = \frac{\sqrt{3}}{2}$.

Based on the values of the sides of the triangle, we now know the coordinates of the point (x, y) where the terminal side of the 60^{o} angle intersects the unit circle. This is the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, as shown below.



We will now repeat this process for a 45^{o} reference angle. We first draw a right triangle that is based on a 45^{o} reference angle, as shown below.

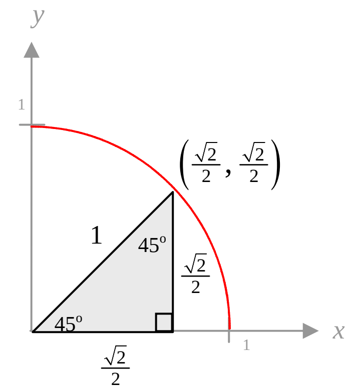


This triangle is a 45^{o}-45^{o}-90^{o} triangle. We again want to find the values of x and y. Recall our theorem from the previous unit:

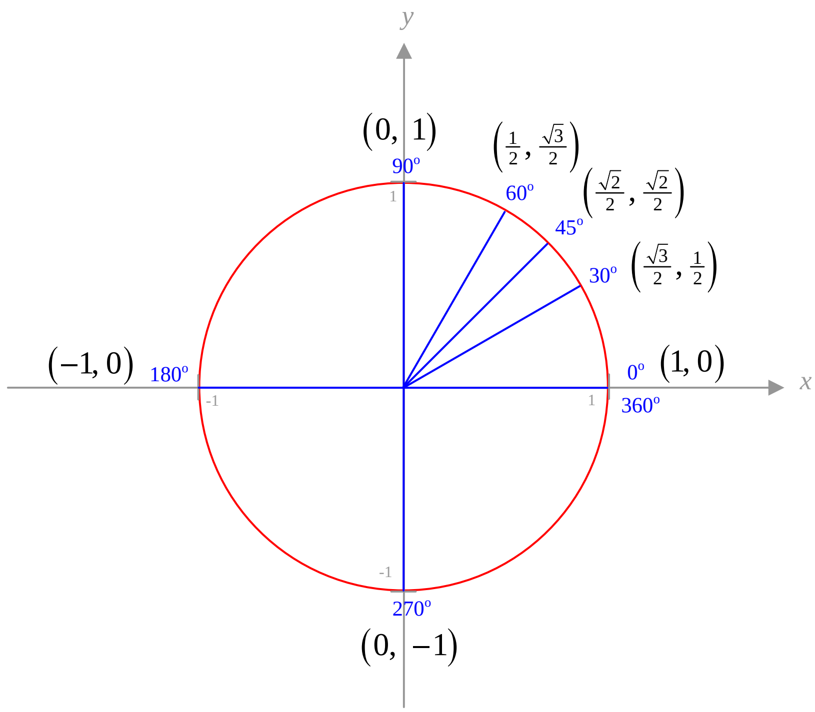
In a 45^{o}-45^{o}-90^{o} triangle, the legs are congruent, and the length of the hypotenuse is $\sqrt{2}$ times the length of either leg.

Since the length of the hypotenuse is $\sqrt{2}$ times the length of either leg, we can say that the hypotenuse has length $x\sqrt{2}$. But we know already that the hypotenuse has length 1, so $x\sqrt{2} = 1$. Solving for x, we find that $x = \frac{1}{\sqrt{2}}$. Rationalizing the denominator, $x = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Since the legs are congruent, $x = y$, so $y = \frac{\sqrt{2}}{2}$.

Based on the values of the sides of the triangle, we now know the coordinates of the point (x, y) where the terminal side of the 45^{o} angle intersects the unit circle. This is the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, as shown below.



Putting together all of the information from this section about quadrantal angles as well as special angles in the first quadrant, we obtain the following diagram:



We will use these coordinates in later sections to find trigonometric functions of special angles on the unit circle.

Definitions of the Six Trigonometric Functions

We will soon learn how to apply the coordinates of the unit circle to find trigonometric functions, but we want to preface this discussion with a more general definition of the six trigonometric functions.

Definitions of the Six Trigonometric Functions: General Case

Let θ be an angle drawn in standard position, and let $P(x, y)$ represent the point where the terminal side of the angle intersects the circle $x^2 + y^2 = r^2$. The six trigonometric functions are defined as follows:

$$\sin(\theta) = \frac{y}{r} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{r}{y} \quad (y \neq 0)$$

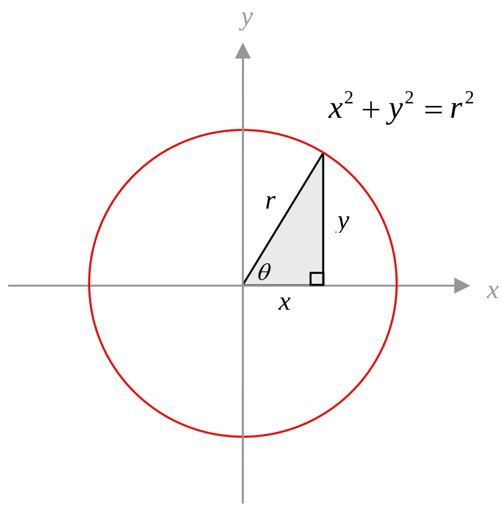
$$\cos(\theta) = \frac{x}{r} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{r}{x} \quad (x \neq 0)$$

$$\tan(\theta) = \frac{y}{x} \quad (x \neq 0) \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{x}{y} \quad (y \neq 0)$$

Alternative definitions for the tangent and the cotangent functions are as follows:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

The above functions are not really new to us if we relate them to our previous unit on right triangle trigonometry. For the purpose of remembering the formulas, we will choose to draw an angle θ in standard position in the first quadrant, and then draw a right triangle in the first quadrant which contains that angle, inscribed in the circle $x^2 + y^2 = r^2$. (Remember that the circle $x^2 + y^2 = r^2$ is centered at the origin with radius *r*.) We label the horizontal side of the triangle *x*, the vertical side *y*, and the hypotenuse *r* (since it represents the radius of the circle.) A diagram of the triangle is shown below.



Notice that the formula $x^2 + y^2 = r^2$ (the equation of the circle) is simply the Pythagorean Theorem as it relates to the sides of the triangle.

Recall the formulas for the basic trigonometric ratios which we learned in the previous unit on right triangle trigonometry, shown below in abbreviated form:

Mnemonic: SOH-CAH-TOA

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$

and the reciprocal trigonometric ratios:

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{Hypotenuse}}{\text{Opposite}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{Hypotenuse}}{\text{Adjacent}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{Adjacent}}{\text{Opposite}}$$

Using these formulas in the triangle from the diagram above, we obtain our six trigonometric functions which we apply to the coordinate plane:

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{y}{r} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{r}{y}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{x}{r} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{r}{x}$$

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{y}{x} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{x}{y}$$

Note that even though we drew the right triangle in the first quadrant in order to easily relate these formulas to right triangle trigonometry, these definitions apply to <u>any</u> angle. (We will discuss later how to properly draw right triangles in other quadrants. Right triangles can not be drawn to illustrate the quadrantal angles, but the above formulas still apply.)

Finally, let us justify the new formulas for tangent and cotangent. (We will do this algebraically, not with our right triangle picture from above.) Our alternative definition for the tangent ratio is $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$. If we substitute $\sin(\theta)=\frac{y}{r}$ and $\cos(\theta)=\frac{x}{r}$ into this equation, we can see that our alternative definition is equivalent to our definition $\tan(\theta)=\frac{y}{x}$. The algebraic justification is shown below:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r}\cdot\frac{r}{x} = \frac{y}{x}$$

Similarly, our alternative definition for the cotangent ratio is $\cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$. If we substitute $\sin(\theta)=\frac{y}{r}$ and $\cos(\theta)=\frac{x}{r}$ into this equation, we can see that our alternative definition for cotangent is equivalent to the formula $\cot(\theta)=\frac{1}{\tan(\theta)}=\frac{x}{y}$. The algebraic justification is shown below:

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r}\cdot\frac{r}{y} = \frac{x}{y}$$

Let us now apply our trigonometric definitions to the unit circle. Since the unit circle has radius 1, we can see that our trigonometric functions are greatly simplified:

$$\sin(\theta)=\frac{y}{r}=\frac{y}{1}=y \qquad \csc(\theta)=\frac{r}{y}=\frac{1}{y},\ y\neq 0$$

$$\cos(\theta)=\frac{x}{r}=\frac{x}{1}=x \qquad \sec(\theta)=\frac{r}{x}=\frac{1}{x},\ x\neq 0$$

$$\tan(\theta)=\frac{y}{x}=\frac{\sin(\theta)}{\cos(\theta)},\ x\neq 0 \qquad \cot(\theta)=\frac{x}{y}=\frac{\cos(\theta)}{\sin(\theta)},\ y\neq 0$$

Definitions of the Six Trigonometric Functions: Special Case of the Unit Circle

Let θ be an angle drawn in standard position, and let $P(x, y)$ represent the point where the terminal side of the angle intersects the unit circle $x^2 + y^2 = 1$. The six trigonometric functions are defined as follows:

$$\sin(\theta) = y \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y} \quad (y \neq 0)$$

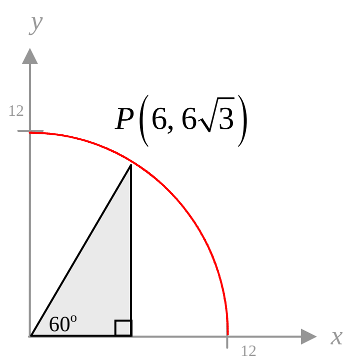
$$\cos(\theta) = x \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x} \quad (x \neq 0)$$

$$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)} \quad (x \neq 0) \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)} \quad (y \neq 0)$$

Note that these definitions apply ONLY to the unit circle!

Note that the value of a trigonometric function for any given angle remains constant regardless of the radius of the circle. For example, let us suppose that we want to find $\sin\left(60^\circ\right)$ in two different ways, using the following two diagrams. (Only the first quadrant is shown, since that is all that is needed for the problem.)

Diagram 1:

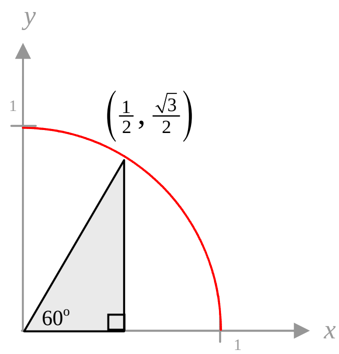


Find $\sin\left(60^\circ\right)$.

Solution:
Since the circle shown has radius 12, we must use the general definition for the sine function, $\sin(\theta) = \frac{y}{r}$. Since the point given is $\left(6, 6\sqrt{3}\right)$, the y value is $6\sqrt{3}$. We know from the diagram that $r = 12$.

Therefore, $\sin\left(60^\circ\right) = \frac{y}{r} = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$.

Diagram 2:



Find $\sin\left(60^\circ\right)$.

Solution:
Since the circle shown has radius 1, we can use the shortened definition of the sine function $\sin(\theta) = y$. Since the point given is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, the y value is $\frac{\sqrt{3}}{2}$.

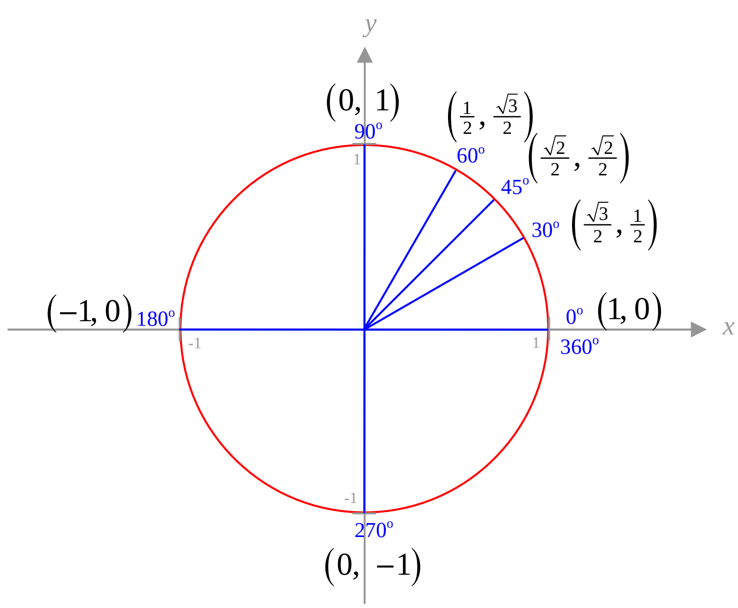
Therefore, $\sin\left(60^\circ\right) = y = \frac{\sqrt{3}}{2}$.

We will use the unit circle (and thus the shortened definitions) to evaluate the trigonometric functions of special angles. (We will see some examples later in the unit where the general definitions of the trigonometric functions can be useful.)

Examples
Find the exact values of the following:

1. $\cos\left(90^\circ\right)$
2. $\sin\left(45^\circ\right)$
3. $\tan\left(30^\circ\right)$
4. $\csc\left(180^\circ\right)$
5. $\sec\left(60^\circ\right)$
6. $\cot\left(0^\circ\right)$
7. $\tan\left(45^\circ\right)$
8. $\sin\left(270^\circ\right)$
9. $\sec\left(30^\circ\right)$

Solutions:
First, recall our unit circle with the coordinates which we have filled in so far:



1. $\cos(90^\circ)$

   The terminal side of a 90^{o} angle intersects the unit circle at the point $(0,1)$.
   Using the definition $\cos(\theta) = x$, we conclude that $\cos(90^\circ) = 0$.

2. $\sin(45^\circ)$

   The terminal side of a 45^{o} angle intersects the unit circle at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
   Using the definition $\sin(\theta) = y$, we conclude that $\sin(45^\circ) = \frac{\sqrt{2}}{2}$.

3. $\tan(30^\circ)$

   The terminal side of a 30^{o} angle intersects the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
   Using the definition $\tan(\theta) = \frac{y}{x}$, we conclude that $\tan(30^\circ) = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$.
   Rationalizing the denominator, $\tan(30^\circ) = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

4. $\csc(180^\circ)$

   The terminal side of a 180^{o} angle intersects the unit circle at the point $(-1, 0)$.
   Using the definition $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y}$, we conclude that $\csc(180^\circ)$ is undefined, since $\frac{1}{0}$ is undefined.

5. $\sec(60^\circ)$

   The terminal side of a 60^{o} angle intersects the unit circle at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.
   Using the definition $\sec(\theta) = \frac{1}{x}$, we conclude that $\sec(60^\circ) = \frac{1}{\frac{1}{2}} = 1 \cdot \frac{2}{1} = 2$.

6. $\cot(0^\circ)$

   The terminal side of the 0^{o} angle intersects the unit circle at the point $(1, 0)$.
   Using the definition $\cot(\theta) = \frac{x}{y}$, we conclude that $\cot(0^\circ)$ is undefined, since $\frac{1}{0}$ is undefined.

7. $\tan\left(45^\circ\right)$

The terminal side of a 45$^{\text{o}}$ angle intersects the unit circle at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Using the definition $\tan\left(\theta\right) = \frac{y}{x}$, we conclude that $\tan\left(45^\circ\right) = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{2}} = 1$.

8. $\sin\left(270^\circ\right)$

The terminal side of a 270$^{\text{o}}$ angle intersects the unit circle at the point $\left(0, -1\right)$.

Using the definition $\sin\left(\theta\right) = y$, we conclude that $\sin\left(270^\circ\right) = -1$.

9. $\sec\left(30^\circ\right)$

The terminal side of a 30$^{\text{o}}$ angle intersects the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Using the definition $\sec\left(\theta\right) = \frac{1}{\cos(\theta)} = \frac{1}{x}$, we conclude that

$\sec\left(30^\circ\right) = \frac{1}{\frac{\sqrt{3}}{2}} = 1 \cdot \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$. Rationalizing the denominator,

$\sec\left(30^\circ\right) = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Example

Find $\tan\left(90^\circ\right)$ and $\cot\left(90^\circ\right)$.

<u>Solution:</u>

The terminal side of a 90$^{\text{o}}$ angle intersects the unit circle at the point $\left(0, 1\right)$.

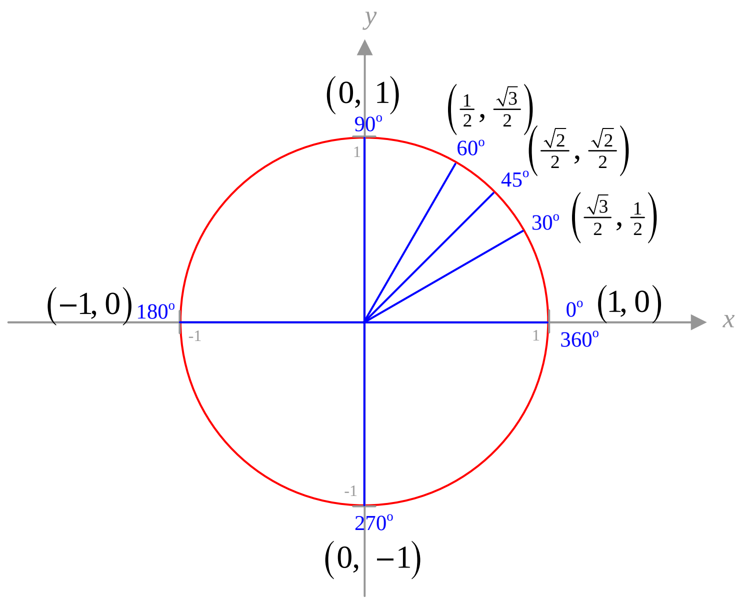
To find $\tan\left(90^\circ\right)$, we use the definition $\tan\left(\theta\right) = \frac{y}{x}$ and conclude that $\tan\left(90^\circ\right)$ is undefined, since $\frac{1}{0}$ is undefined.

To find $\cot\left(90^\circ\right)$, we use the definition $\cot\left(\theta\right) = \frac{x}{y}$ and conclude that $\cot\left(90^\circ\right) = \frac{0}{1} = 0$.

Important Note: If we want to find $\cot\left(90^\circ\right)$ and we instead use the definition $\cot\left(\theta\right) = \frac{1}{\tan(\theta)}$, knowing that $\tan\left(90^\circ\right)$ is undefined, we cannot compute $\frac{1}{\text{undefined}}$. In this case, we need to instead use the direct formula $\cot\left(\theta\right) = \frac{x}{y}$ as we have done above.

Coordinates of All Special Angles on the Unit Circle

We will now label the coordinates of all of the special angles in the unit circle so that we can apply the trigonometric functions to angles in any quadrant. First, recall our unit circle with the coordinates we have filled in so far.

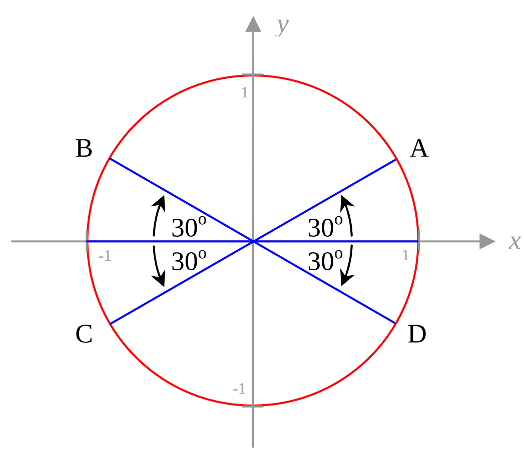


We will now draw all of the angles (between 0^{o} and 360^{o}) that have a 30^{o} reference angle.

> Definition of Reference Angle
>
> When an angle is drawn in standard position, its reference angle is the positive acute angle measured from *x*-axis to the angle's terminal side.

Angles with Reference angles of 30^{o}:

Points A, B, C, and D each represent the point where the terminal side of an angle intersects the unit circle. For each of these points, we wish to determine the angle measure (in standard position from 0^{o} to 360^{o}) as well as the coordinates of that particular angle.



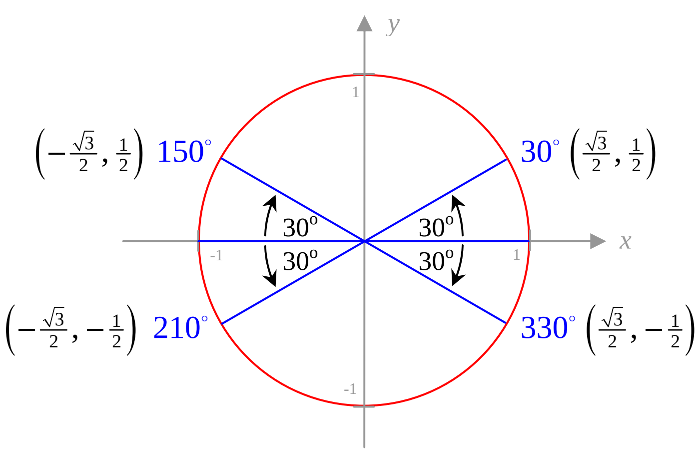
Point A is located along the terminal side of a 30° angle. This point should already be familiar to us; the coordinates of point A are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Point B is located along the terminal side of a 150° angle, since $180^\circ - 30^\circ = 150^\circ$. The coordinates of point B are almost identical to those of point A, except that the *x*-value is negative in the second quadrant, so the coordinates of point B are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Point C is located along the terminal side of a 210° angle, since $180^\circ + 30^\circ = 210^\circ$. The coordinates of point C are almost identical to those of point A, except that the *x*-value and *y*-value are negative in the third quadrant, so the coordinates of point C are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

Point D is located along the terminal side of a 330° angle, since $360^\circ - 30^\circ = 330^\circ$. The coordinates of point D are almost identical to those of point A, except that the *y*-value is negative in the fourth quadrant, so the coordinates of point D are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

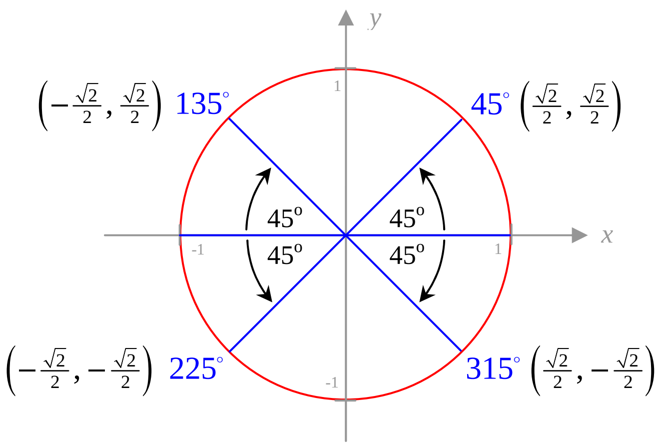
Below is a diagram with all of the 30° reference angles between 0° and 360°, along with their respective coordinates:



We have gone into great detail describing the justification for the angles and coordinates related to 30° reference angles. Since the explanations for the 45° and 60° reference angles are so similar, we will skip the details and simply give the final diagrams with the angles and coordinates labeled.

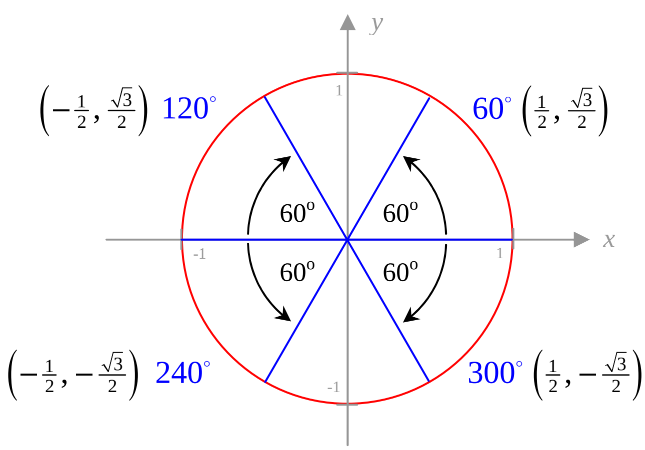
Angles with Reference angles of 45^o:

Below is a diagram with all of the 45^o reference angles between 0^o and 360^o, along with their respective coordinates:



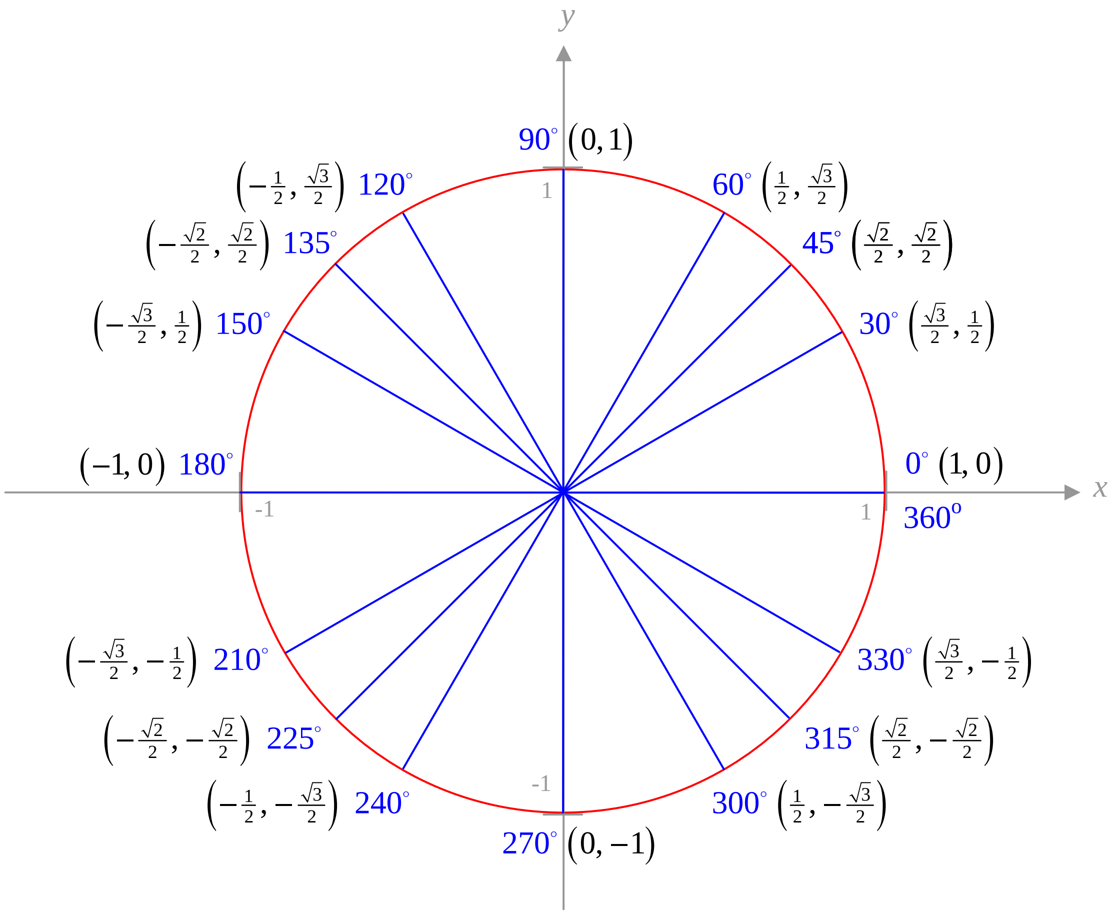
Angles with Reference angles of 60^o:

Below is a diagram with all of the 60^o reference angles between 0^o and 360^o, along with their respective coordinates:



We will now put all of the previous information together in one diagram which includes the coordinates of all special angles on the unit circle.

Coordinates of All Special Angles on the Unit Circle



Evaluating Trigonometric Functions of Any Special Angle

Now that we know the coordinates of all the special angles on the unit circle, we can use these coordinates to find trigonometric functions of any special angle (i.e. any multiple of 30^{o}, 45^{o}, or 60^{o}). First, let us review the concept of coterminal angles.

<u>Definition of Coterminal Angles</u>

<u>Coterminal Angles</u> are angles drawn in standard position that share a terminal side.

For any angle θ, an angle coterminal with θ can be obtained by using the formula $\theta + k \cdot (360^{\circ})$, where k is any integer.

We already have discussed the fact that a 0^o angle is coterminal with a 360^o angle. Below are more examples of coterminal angles.

Example
Find four angles that are coterminal with a 30^o angle drawn in standard position.

> <u>Solution:</u>
> To find angles coterminal with 30^o, we add to it any multiple of 360^o, since adding 360^o means adding exactly one revolution, which keeps the terminal side of the angle in the same position. There are an infinite number of coterminal angles for any given angle, so the following solutions are not unique.
>
> $$30^\circ + 1\cdot\left(360^\circ\right) = 390^\circ$$
> $$30^\circ + 2\cdot\left(360^\circ\right) = 750^\circ$$
> $$30^\circ + (-1)\cdot\left(360^\circ\right) = -330^\circ$$
> $$30^\circ + (-2)\cdot\left(360^\circ\right) = -690^\circ$$
>
> Therefore, four angles coterminal with 30^o are 390^o, 750^o, -330^o, and -690^o.

Since coterminal angles share the same terminal side, they intersect the same point on the unit circle. Therefore, since 30^o intersects the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, the angles 390^o, 750^o, -330^o, and -690^o also intersect the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. This allows us to find trigonometric functions of special angles other than those that we have drawn on our unit circle between 0^o and 360^o.

Next, recall the definitions of the trigonometric functions as they apply to the unit circle:

$$\sin(\theta) = y \qquad \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y} \quad (y \neq 0)$$

$$\cos(\theta) = x \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x} \quad (x \neq 0)$$

$$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)} \quad (x \neq 0) \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)} \quad (y \neq 0)$$

We can now use these definitions to find trigonometric functions of any special angle.

Examples

Find the exact values of the following:

1. $\sin\left(240^\circ\right)$
2. $\cos\left(315^\circ\right)$
3. $\cot\left(210^\circ\right)$
4. $\sec\left(30^\circ\right)$
5. $\tan\left(135^\circ\right)$
6. $\csc\left(630^\circ\right)$
7. $\sin\left(-240^\circ\right)$
8. $\tan\left(-150^\circ\right)$
9. $\sec\left(810^\circ\right)$

<u>Solutions:</u>

1. $\sin\left(240^\circ\right)$

   The terminal side of a 240$^\text{o}$ angle intersects the unit circle at the point $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Using the definition $\sin(\theta) = y$, we conclude that $\sin\left(240^\circ\right) = -\frac{\sqrt{3}}{2}$.

2. $\cos\left(315^\circ\right)$

   The terminal side of a 315$^\text{o}$ angle intersects the unit circle at the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Using the definition $\cos(\theta) = x$, we conclude that $\cos\left(315^\circ\right) = \frac{\sqrt{2}}{2}$.

3. $\cot\left(210^\circ\right)$

   The terminal side of a 210$^\text{o}$ angle intersects the unit circle at the point $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Using the definition $\cot(\theta) = \frac{x}{y}$, we conclude that

   $\cot\left(210^\circ\right) = \frac{-\frac{\sqrt{3}}{2}}{\frac{-1}{2}} = \frac{-\sqrt{3}}{2} \cdot \frac{-2}{1} = \sqrt{3}$. .

4. $\sec\left(30^\circ\right)$

   The terminal side of a 30$^\text{o}$ angle intersects the unit circle at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Using the definition $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x}$, we conclude that

   $\sec\left(30^\circ\right) = \frac{1}{\frac{\sqrt{3}}{2}} = 1 \cdot \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$. Rationalizing the denominator,

   $\sec\left(30^\circ\right) = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

5. $\tan\left(135^\circ\right)$

The terminal side of a 135^o angle intersects the unit circle at the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Using the definition $\tan\left(\theta\right) = \frac{y}{x}$, we conclude that

$\tan\left(135^\circ\right) = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \cdot \frac{-2}{\sqrt{2}} = -1$.

6. $\csc\left(630^\circ\right)$

A 630^o angle is coterminal with a 270^o angle (since $630^\circ - 360^\circ = 270^\circ$) and its terminal side intersects the unit circle at the point $\left(0, -1\right)$. Using the definition $\csc\left(\theta\right) = \frac{1}{\sin(\theta)} = \frac{1}{y}$, we conclude that $\csc\left(630^\circ\right) = \frac{1}{-1} = -1$.

7. $\sin\left(-240^\circ\right)$

A -240^o angle is coterminal with a 120^o angle (since $-240^\circ + 360^\circ = 120^\circ$) and its terminal side intersects the unit circle at the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Using the definition $\sin\left(\theta\right) = y$, we conclude that $\sin\left(-240^\circ\right) = \frac{\sqrt{3}}{2}$.

8. $\tan\left(-150^\circ\right)$

A -150^o angle is coterminal with a 210^o angle (since $-150^\circ + 360^\circ = 210^\circ$) and its terminal side intersects the unit circle at the point $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Using the definition $\tan\left(\theta\right) = \frac{y}{x}$, we conclude that $\tan\left(-150^\circ\right) = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{-1}{2} \cdot \frac{-2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$.

Rationalizing the denominator, we obtain $\tan\left(-150^\circ\right) = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

9. $\sec\left(810^\circ\right)$

A 810^o angle is coterminal with a 90^o angle (since $810^\circ - 2\cdot\left(360^\circ\right) = 90^\circ$) and its terminal side intersects the unit circle at the point $\left(0, 1\right)$. Using the definition $\sec\left(\theta\right) = \frac{1}{\cos(\theta)} = \frac{1}{x}$, we conclude that $\sec\left(810^\circ\right)$ is undefined, since $\frac{1}{0}$ is undefined.

Exercises

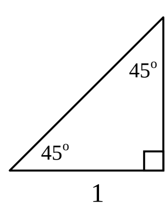
Find the exact values of the following:

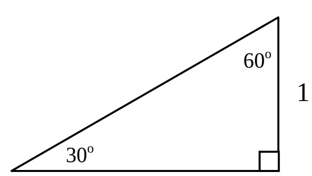
1. $\cos\left(150^\circ\right)$
2. $\tan\left(330^\circ\right)$
3. $\sin\left(225^\circ\right)$
4. $\cot\left(60^\circ\right)$
5. $\csc\left(225^\circ\right)$
6. $\tan\left(-120^\circ\right)$
7. $\cot\left(-450^\circ\right)$
8. $\cos\left(495^\circ\right)$
9. $\sin\left(3630^\circ\right)$

Methods of Finding Trigonometric Functions of Special Angles

We can see that the unit circle assists us greatly in finding trigonometric functions of special angles. But what if we don't have a diagram of the unit circle, and we don't wish to draw one? It is assumed that the quadrantal angles (multiples of 90^o -- on the axes) are fairly easy to visualize without a diagram. One option is to memorize the coordinates of the special angles in the first quadrant of the unit circle (30^o, 45^o, and 60^o), and use those values to find the trigonometric functions of angles in other quadrants. We will discuss a few ways of finding the basic trigonometric functions of 30^o, 45^o, and 60^o (other than memorizing the coordinates of these special angles in the first quadrant of the unit circle).

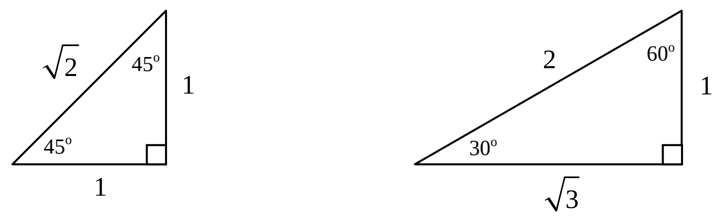
One method is to use a basic 45^o-45^o-90^o triangle and a 30^o-60^o-90^o triangle to derive the trigonometric functions of 30^o, 45^o, and 60^o. We learned earlier in the unit that the trigonometric functions are constant for any given angle; for example, $\sin\left(30^\circ\right)$ is always $\frac{1}{2}$, regardless of the size of the radius of the circle – or the size of the triangle drawn. So we can choose any 45^o-45^o-90^o triangle and 30^o-60^o-90^o triangle to work from. For simplicity, let us choose both triangles to have a shorter leg of length 1, as shown below:





Recall that in a 45^o-45^o-90^o triangle, the legs are congruent, and the length of the hypotenuse is $\sqrt{2}$ times the length of either leg. In a 30^o-60^o-90^o triangle, the length of the hypotenuse is twice the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.

Filling in the missing sides from the diagram above, we obtain the following triangles:



Now recall the trigonometric ratios that we learned for right triangles (shown below in abbreviated form):

SOH-CAH-TOA

$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$ $\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ $\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$

Using these ratios on the triangles above, we obtain the following:

$\sin(45^\circ) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ (Note: This is the y-coordinate of 45° on the unit circle.)

$\cos(45^\circ) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ (Note: This is the x-coordinate of 45° on the unit circle.)

$\tan(45^\circ) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{1}{1} = 1$

$\sin(30^\circ) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{1}{2}$ (Note: This is the y-coordinate of 30° on the unit circle.)

$\cos(30^\circ) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{\sqrt{3}}{2}$ (Note: This is the x-coordinate of 30° on the unit circle.)

$\tan(30^\circ) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin(60^\circ) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\sqrt{3}}{2}$ (Note: This is the y-coordinate of 60° on the unit circle.)

$\cos(60^\circ) = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{1}{2}$ (Note: This is the x-coordinate of 60° on the unit circle.)

$\tan(60^\circ) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\sqrt{3}}{1}$

The reciprocal trigonometric ratios cosecant, secant, and cotangent can be obtained by simply taking the reciprocals of the sine, cosine, and tangent ratios above.

Another method of remembering the trigonometric functions of 30^o, 45^o, and 60^o, along with 0^o and 90^o, is shown below. (This is more of a 'trick' for remembering the ratios, so no mathematical justification is given.)

<u>Step 1:</u>
Label the columns at the top of the chart for all of the special angles between 0^o and 90^o in ascending order, as shown. Also label the rows with the words "sine" and "cosine."

	0^o	**30^o**	**45^o**	**60^o**	**90^o**
sine					
cosine					

<u>Step 2:</u> *This chart is not yet complete!*
Write the numbers 0, 1, 2, 3, 4 in the "sine" row, and the numbers 4, 3, 2, 1, 0 in the "cosine" row, as shown.

	0^o	**30^o**	**45^o**	**60^o**	**90^o**
sine	0	1	2	3	4
cosine	4	3	2	1	0

This chart is not yet complete!

This chart is not yet complete!

<u>Step 3:</u> *This chart is correct, but not yet simplified…*
For each of the numbers in the "sine" and "cosine" rows, take the square root of the number and then divide by 2.

	0^o	**30^o**	**45^o**	**60^o**	**90^o**
sine	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cosine	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

We then simplify each of the numbers in the chart above.

<u>Step 4:</u> ***This is the final version of the chart!***
Simplify each of the values from the table above, and we obtain our final chart:

	0^o	30^o	45^o	60^o	90^o	
sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	⇨ *y*-values on the unit circle
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	⇨ *x*-values on the unit circle

Examples

Use the methods learned in this section (special right triangles or the chart above) to find the exact values of the following trigonometric functions. *Note that these examples were also included in the previous section, but will now be solved using a different method.*

1. $\sin\left(240^\circ\right)$
2. $\cos\left(315^\circ\right)$
3. $\cot\left(210^\circ\right)$
4. $\sec\left(30^\circ\right)$
5. $\tan\left(135^\circ\right)$

<u>Solutions:</u>

1. $\sin\left(240^\circ\right)$

   The terminal side of a 240^o angle measures 60^o to the *x*-axis, so we use a 60^o reference angle. Using either a 30^o-60^o-90^o triangle or our chart from above, we find that $\sin\left(60^\circ\right)=\frac{\sqrt{3}}{2}$. Since a 240^o angle is in the third quadrant, and *y*-values in the third quadrant are negative, $\sin\left(240^\circ\right)$ must be negative. Since $\sin\left(60^\circ\right)=\frac{\sqrt{3}}{2}$, we conclude that $\sin\left(240^\circ\right)=-\frac{\sqrt{3}}{2}$.

2. $\cos\left(315^\circ\right)$

   The terminal side of a 315^o angle measures 45^o to the *x*-axis, so we use a 45^o reference angle. Using either a 45^o-45^o-90^o triangle or our chart from above, we find that $\cos\left(45^\circ\right)=\frac{\sqrt{2}}{2}$. Since a 315^o angle is in the fourth quadrant, and *x*-values in the fourth quadrant are positive, $\cos\left(315^\circ\right)$ must be positive. Since $\cos\left(45^\circ\right)=\frac{\sqrt{2}}{2}$, we conclude that $\cos\left(315^\circ\right)=\frac{\sqrt{2}}{2}$.

3. $\cot\left(210^\circ\right)$

The terminal side of a 210^o angle measures 30^o to the *x*-axis, so we use a 30^o reference angle. Remember that $\cot\left(\theta\right)=\frac{1}{\tan(\theta)}=\frac{\cos(\theta)}{\sin(\theta)}$. If we use the 30^o-60^o-90^o triangle, we find that $\tan\left(30^\circ\right)=\frac{1}{\sqrt{3}}$, so $\cot\left(30^\circ\right)=\frac{1}{\tan(\theta)}=\sqrt{3}$. If we instead use our chart from above, we find that $\cos\left(30^\circ\right)=\frac{\sqrt{3}}{2}$ and $\sin\left(30^\circ\right)=\frac{1}{2}$, so $\cot\left(30^\circ\right)=\frac{\cos\left(30^\circ\right)}{\sin\left(30^\circ\right)}=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\frac{\sqrt{3}}{2}\cdot\frac{2}{1}=\sqrt{3}$. Since a 210^o angle is in the third quadrant, and both the *x* and *y*-values in the third quadrant are negative, this means that both $\cos\left(210^\circ\right)$ and $\sin\left(210^\circ\right)$ are negative, therefore their quotient $\cot\left(210^\circ\right)=\frac{\cos\left(210^\circ\right)}{\sin\left(210^\circ\right)}$ is positive. Since $\cot\left(30^\circ\right)=\sqrt{3}$, we conclude that $\cot\left(210^\circ\right)=\sqrt{3}$.

4. $\sec\left(30^\circ\right)$

The terminal side of a 30^o angle measures 30^o to the *x*-axis, so we use a 30^o reference angle. (A reference angle is not really necessary in this case, since any angle which is located in the first quadrant can be used as-is.) Remember that $\sec\left(\theta\right)=\frac{1}{\cos(\theta)}$. Using either a 30^o-60^o-90^o triangle or our chart from above, we find that $\cos\left(30^\circ\right)=\frac{\sqrt{3}}{2}$, so $\sec\left(30^\circ\right)=\frac{1}{\cos\left(30^\circ\right)}=\frac{1}{\frac{\sqrt{3}}{2}}=1\cdot\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}$.

Rationalizing the denominator, $\sec\left(30^\circ\right)=\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$.

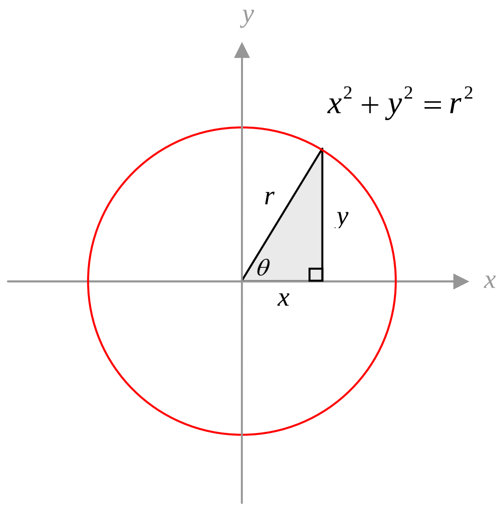
5. $\tan\left(135^\circ\right)$

The terminal side of a 135^o angle measures 45^o to the *x*-axis, so we use a 45^o reference angle. Remember that $\tan\left(\theta\right)=\frac{\sin(\theta)}{\cos(\theta)}$. If we use the 45^o-45^o-90^o triangle, we find that $\tan\left(45^\circ\right)=\frac{1}{1}=1$. If we instead use our chart from above, $\sin\left(45^\circ\right)=\frac{\sqrt{2}}{2}$ and $\sin\left(45^\circ\right)=\frac{\sqrt{2}}{2}$, so $\tan\left(45^\circ\right)=\frac{\sin\left(45^\circ\right)}{\cos\left(45^\circ\right)}=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=\frac{\sqrt{2}}{2}\cdot\frac{2}{\sqrt{2}}=1$. Since a 135^o angle is in the second quadrant, the *x*-value is negative and the *y*-value is positive, which means that $\cos\left(135^\circ\right)$ is negative and $\sin\left(135^\circ\right)$ is positive; therefore their quotient $\tan\left(135^\circ\right)=\frac{\sin\left(135^\circ\right)}{\cos\left(135^\circ\right)}$ is negative. Since $\tan\left(45^\circ\right)=1$, we conclude that $\tan\left(135^\circ\right)=-1$.

Evaluating Trigonometric Functions of Other Angles

Now that we have explored the trigonometric functions of special angles, we will briefly look at how to find trigonometric functions of other angles.

First, we will derive an important trigonometric identity, known as a Pythagorean Identity. It can be related to the Pythagorean Theorem; consider the following right triangle (drawn in the first quadrant for simplicity).



We can see that the equation $x^2 + y^2 = r^2$ (the equation of a circle with radius *r*) is also the Pythagorean Theorem as it relates to the right triangle above.

Let us now direct our attention to the unit circle. Since the radius is 1, any point on the circle itself satisfies the equation $x^2 + y^2 = 1$ (the equation of a circle with radius 1). On the unit circle, we know that $x = \cos(\theta)$ and $y = \sin(\theta)$. Substituting these into the equation $x^2 + y^2 = 1$, we obtain the equation $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. It is standard practice in trigonometry to write these individual terms in shortened form. We write $(\cos(\theta))^2$ as $\cos^2(\theta)$, and we write $(\sin(\theta))^2$ as $\sin^2(\theta)$. We then obtain the following trigonometric identity:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

The identity $\cos^2(\theta) + \sin^2(\theta) = 1$ applies to an angle drawn in any circle of radius *r*, not just the unit circle. A short justification is shown below.

We begin with the general equation of a circle of radius *r*:

$$x^2 + y^2 = r^2$$

Dividing both sides by r^2, we obtain the equation

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}.$$

Simplifying the equation,

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1.$$

The equation can then be rewritten as

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$

For a circle of radius *r*, we know from our general trigonometric definitions that $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$. We substitute these into the above equation and conclude that

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

We will now use this Pythagorean identity to help us to find trigonometric functions of angles.

Example

If $\sin(\theta) = \frac{3}{5}$ and $90^\circ < \theta < 180^\circ$, find the exact values of $\cos(\theta)$ and $\tan(\theta)$.

Solution:

Since $90^\circ < \theta < 180^\circ$, the terminal side of θ is in the second quadrant. Looking at our unit circle, we can easily see that there is no special angle with a *y*-value of $\frac{3}{5}$. There are two methods by which we can solve this problem, both of which are shown below.

Method 1: Our first method is to use the Pythagorean identity

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Since $\sin(\theta) = \frac{3}{5}$, we plug it into the equation

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

$$\cos^2(\theta) + \left(\tfrac{3}{5}\right)^2 = 1$$

We then simplify the equation and solve for $\cos(\theta)$:

$\cos^2(\theta) + \frac{9}{25} = 1$

$\cos^2(\theta) = 1 - \frac{9}{25}$

$\cos^2(\theta) = \frac{25}{25} - \frac{9}{25}$

$\cos^2(\theta) = \frac{16}{25}$

$\cos(\theta) = \frac{4}{5}$ **OR** $\cos(\theta) = -\frac{4}{5}$ (We need to choose; see below...)

$\cos(\theta) = -\frac{4}{5}$ The terminal side of θ is in the second quadrant. We know that x values in the second quadrant are negative, therefore $\cos(\theta)$ is negative.

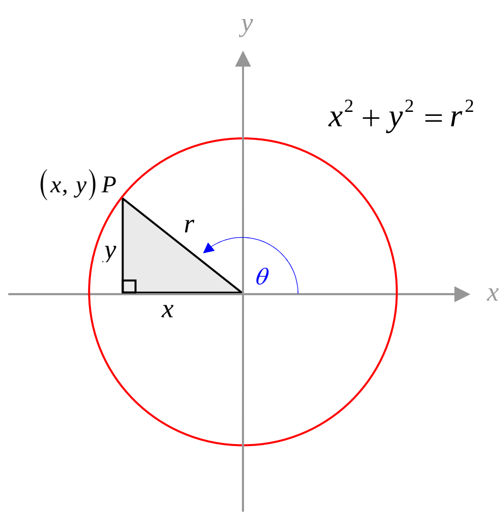
We now want to find $\tan(\theta)$. Using the definition $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$,

$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{5} \cdot -\frac{5}{4} = -\frac{3}{4}$.

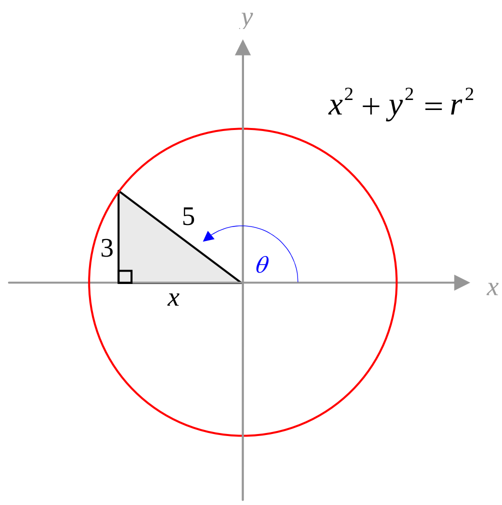
$\tan(\theta) = -\frac{3}{4}$

We conclude that $\cos(\theta) = -\frac{4}{5}$ and $\tan(\theta) = -\frac{3}{4}$.

Method 2: Our second method is to draw a right triangle in a circle of radius r. Since $90^\circ < \theta < 180^\circ$, we draw the right triangle in the second quadrant, as shown below. (When drawing the right triangle, we must make sure that the right triangle has one leg on the x-axis and that one acute angle of the triangle -- the reference angle of θ -- has its vertex at the origin.)



Since $\sin(\theta) = \frac{3}{5}$, and we know that $\sin(\theta) = \frac{y}{r}$, we can label the diagram with the values $y = 3$ and $r = 5$, as shown below. (The formula $\sin(\theta) = \frac{y}{r}$ can be remembered by using the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, using the reference angle of θ which is inside the triangle.) Note that when choosing how to label the sides of the triangle, the radius of the circle is always chosen to be positive.



To find the value of *x*, we then use the equation for the circle (or equivalently, the Pythagorean Theorem) $x^2 + y^2 = r^2$:

$x^2 + 3^2 = 5^2$
$x^2 + 9 = 25$
$x^2 = 16$
$x = 4$ **OR** $x = -4$ (We need to choose; see below.)

$x = -4$ The terminal side of θ is in the second quadrant. We know that *x* values in the second quadrant are negative, therefore we choose the negative value for *x*.

To find $\cos(\theta)$, we now use the formula $\cos(\theta) = \frac{x}{r}$. (The formula $\cos(\theta) = \frac{x}{r}$ can be remembered by using the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$, using the reference angle of θ which is inside the triangle.) Note that we can **NOT** use the formula $\cos(\theta) = x$, since that trigonometric definition only applies to the unit circle, and the circle that we created has radius 5.

$\cos(\theta) = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}$

To find $\tan(\theta)$, we now use the formula $\tan(\theta)=\frac{y}{x}$. (Remember that the formula $\tan(\theta)=\frac{y}{x}$ can be remembered by using the ratio $\frac{\text{opposite}}{\text{adjacent}}$, using the reference angle of θ which is inside the triangle.)

$\tan(\theta)=\frac{y}{x}=\frac{3}{-4}=-\frac{3}{4}$

Note that we could have also used the formula

$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}=\frac{\frac{3}{5}}{\frac{-4}{5}}=\frac{3}{5}\cdot\frac{5}{-4}=-\frac{3}{4}$

We conclude that $\cos(\theta)=-\frac{4}{5}$ and $\tan(\theta)=-\frac{3}{4}$.

We now wish to derive two other Pythagorean identities. Both identities can be easily derived from the identity $\cos^2(\theta)+\sin^2(\theta)=1$.

First, begin with the identity $\cos^2(\theta)+\sin^2(\theta)=1$.

Now divide each term by $\cos^2(\theta)$:

$$\frac{\cos^2(\theta)}{\cos^2(\theta)}+\frac{\sin^2(\theta)}{\cos^2(\theta)}=\frac{1}{\cos^2(\theta)}$$

Since $\frac{\sin(\theta)}{\cos(\theta)}=\tan(\theta)$ and $\frac{1}{\cos(\theta)}=\sec(\theta)$, we can simplify the above equation and obtain the following Pythagorean identity:

$$1+\tan^2(\theta)=\sec^2(\theta)$$

In a similar fashion, we again begin with the identity $\cos^2(\theta)+\sin^2(\theta)=1$.

Now divide each term by $\sin^2(\theta)$:

$$\frac{\cos^2(\theta)}{\sin^2(\theta)}+\frac{\sin^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)}$$

Since $\frac{\cos(\theta)}{\sin(\theta)}=\cot(\theta)$ and $\frac{1}{\sin(\theta)}=\csc(\theta)$, we can simplify the above equation and obtain the following Pythagorean identity:

$$\cot^2(\theta)+1=\csc^2(\theta)$$

Pythagorean Identities

$\cos^2(\theta)+\sin^2(\theta)=1$ $\quad$ $1+\tan^2(\theta)=\sec^2(\theta)$ $\quad$ $\cot^2(\theta)+1=\csc^2(\theta)$

Example

If $\cot(\theta)=\frac{\sqrt{11}}{5}$ and $90^\circ<\theta<270^\circ$, find the exact values of $\sin(\theta)$ and $\cos(\theta)$.

Solution:

Since $90^\circ<\theta<270^\circ$, the terminal side of θ is either in the second or the third quadrant. Recall the definition for cotangent $\cot(\theta)=\frac{x}{y}$; we are given that $\cot(\theta)=\frac{\sqrt{11}}{5}$, which is a positive number. In the second quadrant, *x* is negative and *y* is positive, so the quotient $\cot(\theta)=\frac{x}{y}$ is negative; therefore the terminal side of θ can not lie in the second quadrant. In the third quadrant, on the other hand, both *x* and *y* are negative, so the quotient $\cot(\theta)=\frac{x}{y}$ is positive, which is consistent with the given information. We then conclude that the terminal side of θ is in the third quadrant. There are two methods by which we can solve this problem, both of which are shown below.

Method 1: Our first method is to use the Pythagorean identity $\cot^2(\theta)+1=\csc^2(\theta)$.

Since $\cot(\theta)=\frac{\sqrt{11}}{5}$, we plug it into the equation $\cot^2(\theta)+1=\csc^2(\theta)$.

$$\left(\frac{\sqrt{11}}{5}\right)^2+1=\csc^2(\theta)$$

We then simplify the equation and solve for $\csc(\theta)$:

$$\left(\frac{\sqrt{11}}{5}\right)^2+1=\csc^2(\theta)$$
$$\frac{11}{25}+1=\csc^2(\theta)$$
$$\frac{11}{25}+\frac{25}{25}=\csc^2(\theta)$$
$$\frac{36}{25}=\csc^2(\theta)$$

$\csc(\theta) = \frac{6}{5}$ **OR** $\csc(\theta) = -\frac{6}{5}$ (We need to choose; see below.)

$\csc(\theta) = -\frac{6}{5}$ The terminal side of θ is in the third quadrant. We know that *y* values in the third quadrant are negative, which means that $\sin(\theta)$ is negative. Since $\csc(\theta) = \frac{1}{\sin(\theta)}$, we conclude that $\csc(\theta)$ is negative.

We now want to find $\sin(\theta)$. Since the cosecant and sine functions are reciprocals of each other, we can rearrange the equation $\csc(\theta) = \frac{1}{\sin(\theta)}$ to say that $\sin(\theta) = \frac{1}{\csc(\theta)}$.

$$\sin(\theta) = \frac{1}{\csc(\theta)} = \frac{1}{-\frac{6}{5}} = 1 \cdot \frac{-5}{6} = -\frac{5}{6}.$$

Our final step is to find $\cos(\theta)$. Using the identity

$\cos^2(\theta) + \sin^2(\theta) = 1$,

$\cos^2(\theta) + \left(-\frac{5}{6}\right)^2 = 1$

$\cos^2(\theta) + \frac{25}{36} = 1$

$\cos^2(\theta) = 1 - \frac{25}{36} = \frac{36}{36} - \frac{25}{36}$

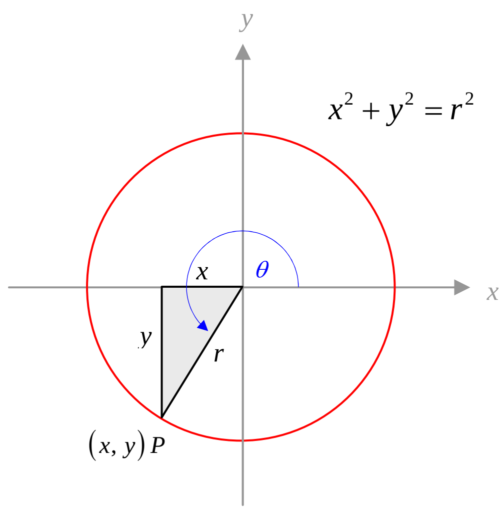
$\cos^2(\theta) = \frac{11}{36}$

$\cos(\theta) = \frac{\sqrt{11}}{6}$ **OR** $\cos(\theta) = -\frac{\sqrt{11}}{6}$ (We need to choose; see below.)

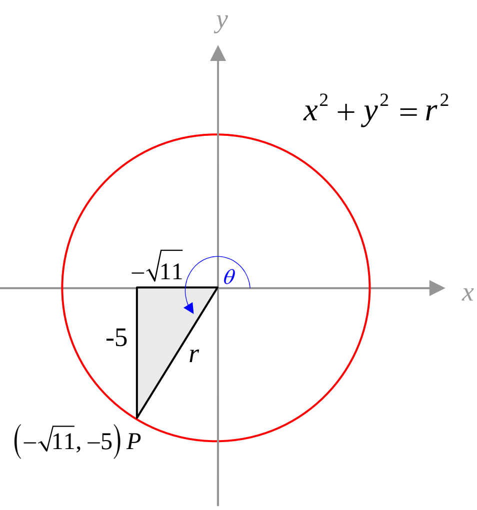
$\cos(\theta) = -\frac{\sqrt{11}}{6}$ The terminal side of θ is in the third quadrant. We know that *x* values in the third quadrant are negative, which means that $\cos(\theta)$ is negative.

We conclude that $\sin(\theta) = -\frac{5}{6}$ and $\cos(\theta) = -\frac{\sqrt{11}}{6}$.

Method 2: Our second method is to draw a right triangle in a circle of radius *r*. Since we have determined that the terminal side of θ is in the third quadrant, we draw the right triangle in the third quadrant, as shown below. (When drawing the right triangle, we must make sure that the right triangle has one leg on the *x*-axis and that one acute angle of the triangle -- the reference angle of θ -- has its vertex at the origin.)



Since $\cot(\theta) = \frac{\sqrt{11}}{5}$, and we know that $\cot(\theta) = \frac{x}{y}$, we can label the diagram with the values $x = -\sqrt{11}$ and $y = -5$, as shown below. We can NOT label the sides of the triangle with the values $x = \sqrt{11}$ and $y = 5$, since the triangle is in the third quadrant, where both *x* and *y* are negative. (The formula $\cot(\theta) = \frac{x}{y}$ can be remembered by using the ratio $\frac{\text{adjacent}}{\text{opposite}}$, using the reference angle of θ which is inside the triangle.)



To find the value of *x*, we then use the equation for the circle (or equivalently, the Pythagorean Theorem) $x^2 + y^2 = r^2$:

$\left(-\sqrt{11}\right)^2 + (-5)^2 = r^2$, so $11 + 25 = r^2$

$36 = r^2$

$r = 6$ (Note that the radius of the circle is always positive.)

To find $\sin(\theta)$, we now use the formula $\sin(\theta) = \frac{y}{r}$. (The formula $\sin(\theta) = \frac{y}{r}$ can be remembered by using the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, using the reference angle of θ which is inside the triangle.) Note that we can **NOT** use the formula $\sin(\theta) = y$, since that trigonometric definition only applies to the unit circle, and the circle that we created has radius 6.

$$\sin(\theta) = \frac{y}{r} = \frac{-5}{6} = -\frac{5}{6}$$

To find $\cos(\theta)$, we now use the formula $\cos(\theta) = \frac{x}{r}$. (The formula $\cos(\theta) = \frac{x}{r}$ can be remembered by using the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$, using the reference angle of θ which is inside the triangle.) Note that we can **NOT** use the formula $\cos(\theta) = x$, since that trigonometric definition only applies to the unit circle, and the circle that we created has radius 6.

$$\cos(\theta) = \frac{x}{r} = \frac{-\sqrt{11}}{6} = -\frac{\sqrt{11}}{6}$$

We conclude that $\sin(\theta) = -\frac{5}{6}$ and $\cos(\theta) = -\frac{\sqrt{11}}{6}$.

Exercises

Answer the following, using either of the two methods described in this section.

1. If $\cos(\theta) = -\frac{5}{13}$ and $180^\circ < \theta < 360^\circ$, find the exact values of $\sin(\theta)$ and $\tan(\theta)$.

2. If $\csc(\theta) = \frac{25}{7}$ and $90^\circ < \theta < 270^\circ$, find the exact values of $\cos(\theta)$ and $\cot(\theta)$.

3. If $\tan(\theta) = -\frac{\sqrt{5}}{2}$ and $180^\circ < \theta < 360^\circ$, find the exact values of $\cos(\theta)$ and $\sin(\theta)$.

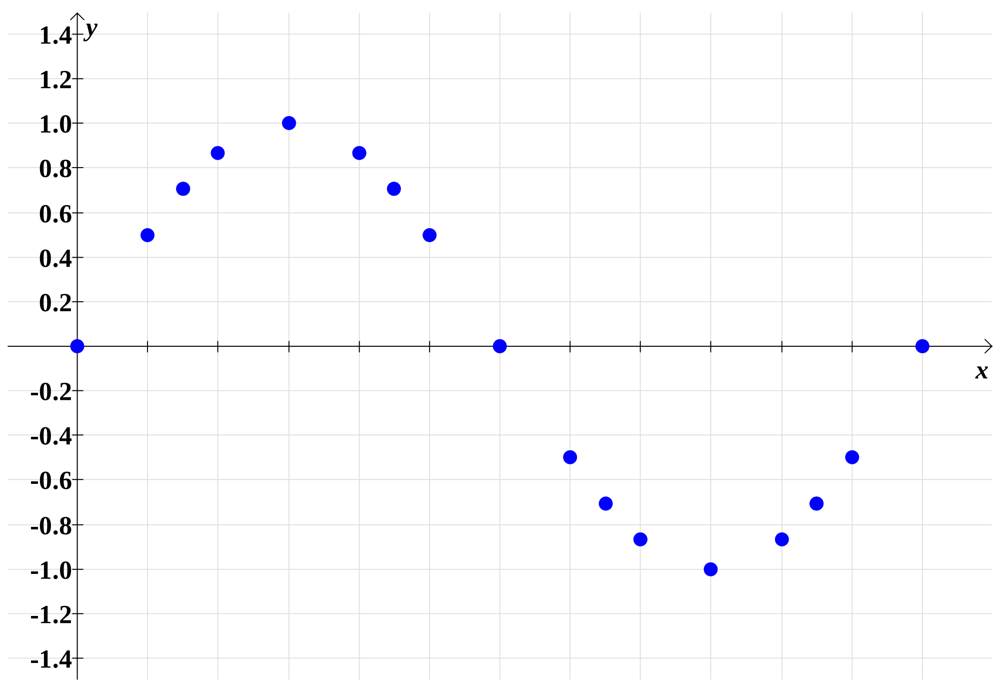
Graphs of the Sine and Cosine Functions

In this section, we will learn how to graph the sine and cosine functions. To do this, we will once again use the coordinates of the special angles from the unit circle.

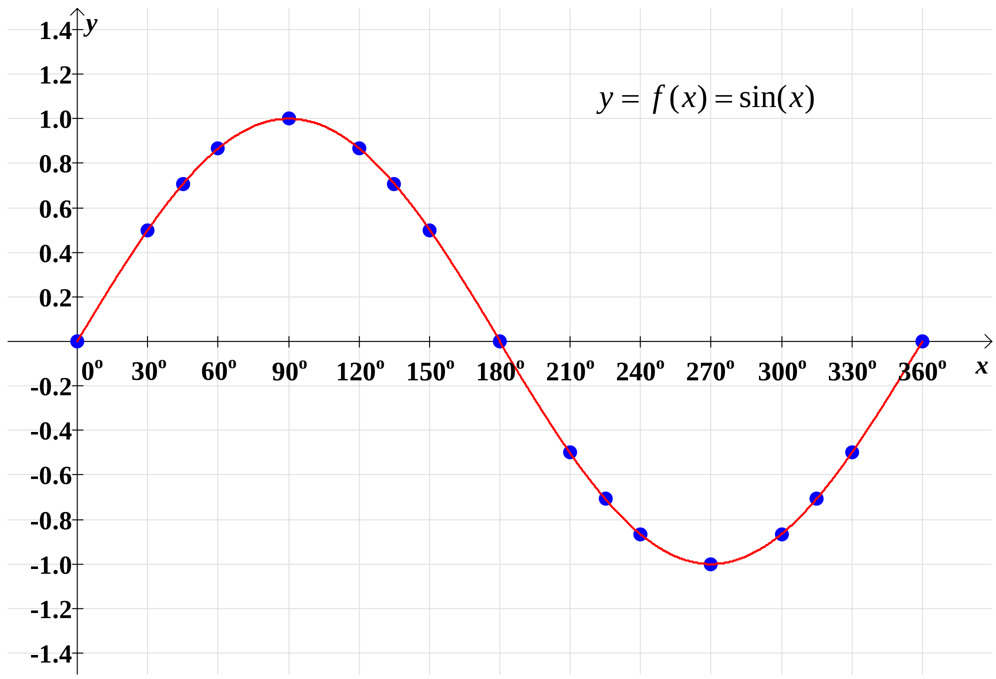
We will first make a chart of values for $y = f(x) = \sin(x)$, where x represents the degree measure of the angle. In the column for the y values, the exact value has also been written as a decimal, rounded to the nearest hundredth for graphing purposes.

x	*y = sin(x)*		*x*	*y = sin(x)*
0^o	0		180^o	0
30^o	$\frac{1}{2} = 0.5$		210^o	$-\frac{1}{2} = -0.5$
45^o	$\frac{\sqrt{2}}{2} \approx 0.71$		225^o	$-\frac{\sqrt{2}}{2} \approx -0.71$
60^o	$\frac{\sqrt{3}}{2} \approx 0.87$		240^o	$-\frac{\sqrt{3}}{2} \approx -0.87$
90^o	1		270^o	-1
120^o	$\frac{\sqrt{3}}{2} \approx 0.87$		300^o	$-\frac{\sqrt{3}}{2} \approx -0.87$
135^o	$\frac{\sqrt{2}}{2} \approx 0.71$		315^o	$-\frac{\sqrt{2}}{2} \approx -0.71$
150^o	$\frac{1}{2} = 0.5$		330^o	$-\frac{1}{2} = -0.5$
			360^o	0

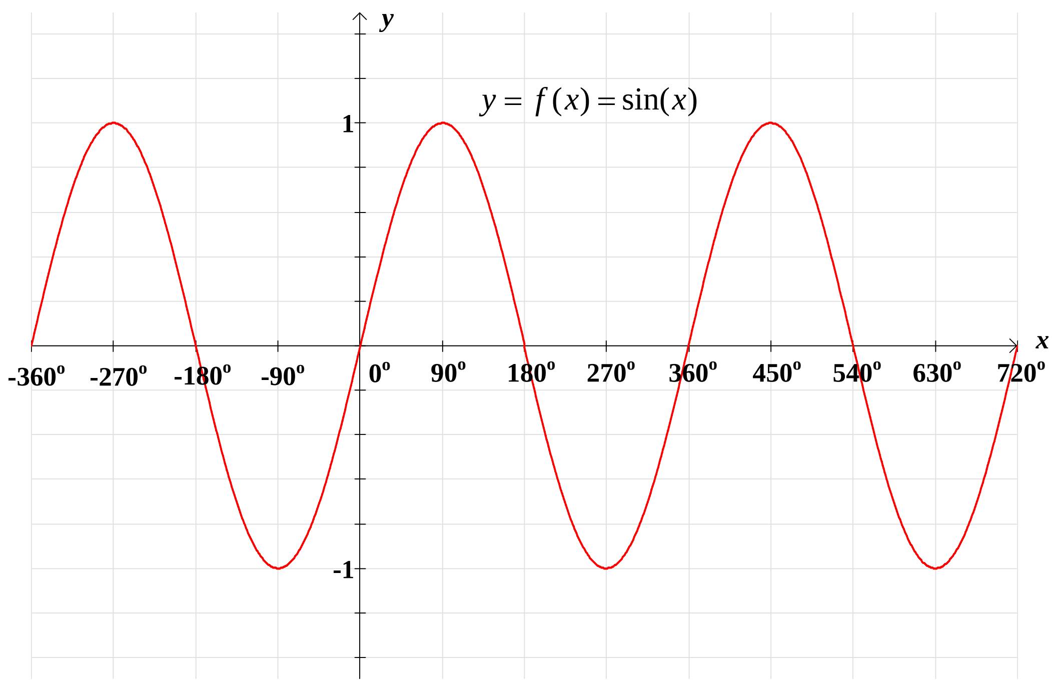
We now plot the above x and y values on the coordinate plane, as shown:



Drawing a smooth curve through the points which we have plotted, we obtain the following graph of $y = f(x) = \sin(x)$:



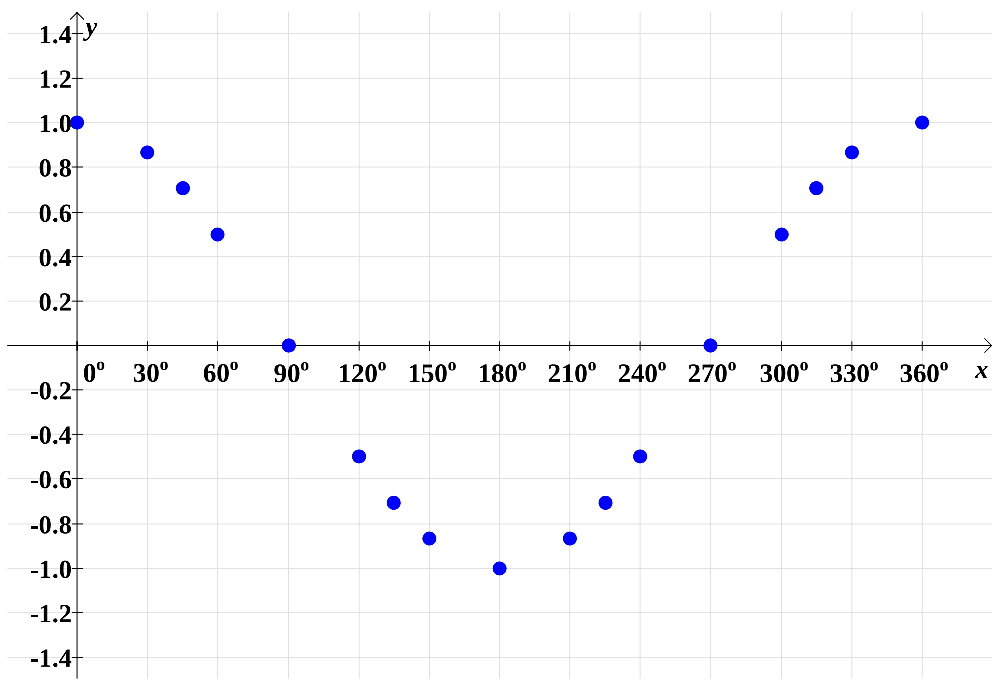
Since 360º is coterminal with 0º, their *y*-values are the same. This is the case with any coterminal angles; 450º is coterminal with 90º, 540º is coterminal with 180º, etc. For this reason, the above graph will repeat itself over and over again, as shown below:



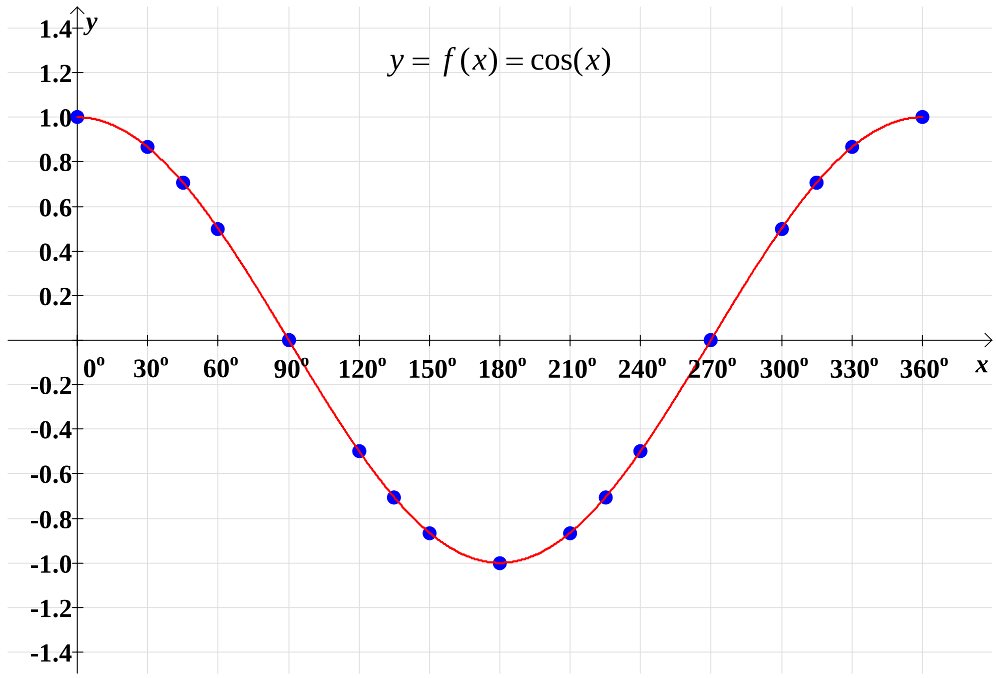
We will now repeat the same process to graph the cosine function. First, we will make a chart of values for $y = f(x) = \cos(x)$, where x represents the degree measure of the angle. In the column for the y values, the exact value has also been written as a decimal, rounded to the nearest hundredth for graphing purposes.

x	***y = cos(x)***		***x***	***y = cos(x)***
0^{o}	1		180^{o}	-1
30^{o}	$\frac{\sqrt{3}}{2} \approx 0.87$		210^{o}	$-\frac{\sqrt{3}}{2} \approx -0.87$
45^{o}	$\frac{\sqrt{2}}{2} \approx 0.71$		225^{o}	$-\frac{\sqrt{2}}{2} \approx -0.71$
60^{o}	$\frac{1}{2} = 0.5$		240^{o}	$-\frac{1}{2} = -0.5$
90^{o}	0		270^{o}	0
120^{o}	$-\frac{1}{2} = -0.5$		300^{o}	$\frac{1}{2} = 0.5$
135^{o}	$-\frac{\sqrt{2}}{2} \approx -0.71$		315^{o}	$\frac{\sqrt{2}}{2} \approx 0.71$
150^{o}	$-\frac{\sqrt{3}}{2} \approx -0.87$		330^{o}	$\frac{\sqrt{3}}{2} \approx 0.87$
			360^{o}	1

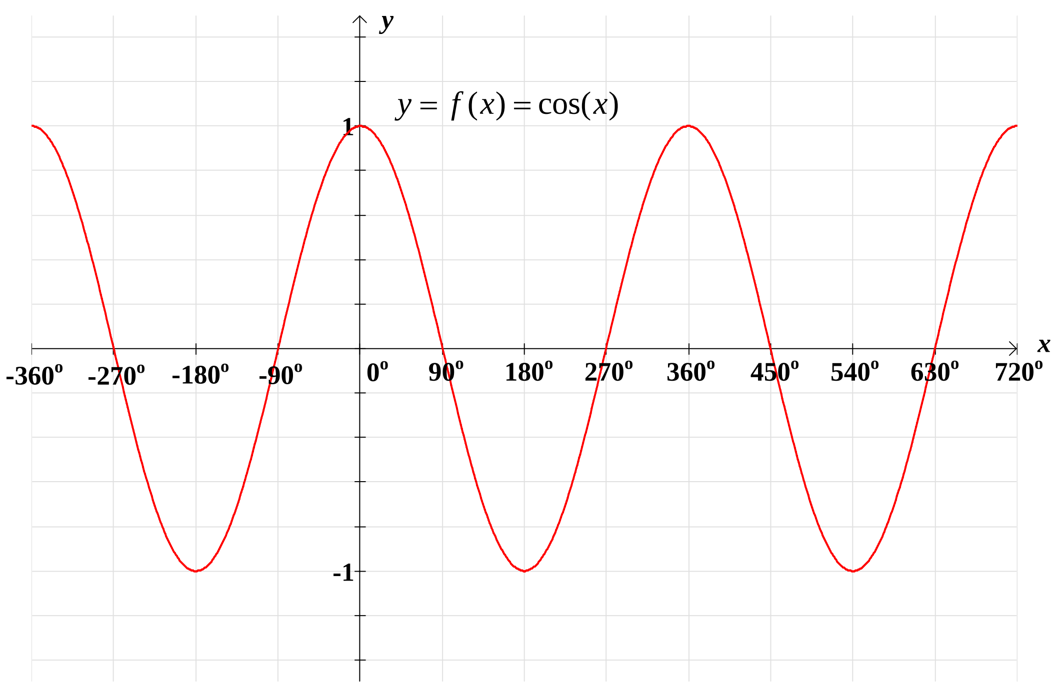
We now plot the above x and y values on the coordinate plane, as shown:



Drawing a smooth curve through the points which we have plotted, we obtain the following graph of $y = f(x) = \cos(x)$:



Since coterminal angles occur every 360°, the cosine graph will repeat itself over and over again, as shown below:



The graphs can easily be used to determine the sine, cosine, secant, and cosecant of quadrantal angles.

Examples
Use the graphs of the sine and cosine functions to find exact values of the following:

1. $\sin\left(90^\circ\right)$

   Solution:
   If we look at the graph of $y = \sin(x)$, where $x = 90^\circ$, we find that the *y*-value is 1. Therefore, we conclude that $\sin\left(90^\circ\right) = 1$.

2. $\cos\left(270^\circ\right)$
   Solution:
   If we look at the graph of $y = \cos(x)$, where $x = 270^\circ$, we find that the *y*-value is 0. Therefore, we conclude that $\cos\left(270^\circ\right) = 0$.

3. $\sec\left(180^\circ\right)$
   Solution:
   If we look at the graph of $y = \cos(x)$, where $x = 180^\circ$, we find that the *y*-value is -1. This means that $\cos\left(180^\circ\right) = -1$. Since $\sec\left(\theta\right) = \frac{1}{\cos(\theta)}$, we conclude that $\sec\left(180^\circ\right) = \frac{1}{\cos\left(180^\circ\right)} = \frac{1}{-1} = -1$.

4. $\csc\left(360^\circ\right)$
   Solution:
   If we look at the graph of $y = \sin(x)$, where $x = 360^\circ$, we find that the *y*-value is 0. This means that $\sin(360^\circ) = 0$. Since $\csc\left(\theta\right) = \frac{1}{\sin(\theta)}$, we conclude that $\csc\left(360^\circ\right)$ is undefined, since $\frac{1}{0}$ is undefined.

Exercises
1. Use the graphs of the sine and cosine functions to find exact values of the following:
   a) $\cos\left(270^\circ\right)$ b) $\sin\left(-90^\circ\right)$ c) $\sec\left(90^\circ\right)$ d) $\csc\left(450^\circ\right)$

2. Sketch the graphs of the following functions. Label all intercepts.
   a) $y = f(x) = \sin(x)$, where $-90^\circ \le x \le 270^\circ$
   b) $y = f(x) = \cos(x)$, where $-720^\circ \le x \le 90^\circ$