Lesson 18.2: Right Triangle Trigonometry

Although Trigonometry is used to solve many problems, historically it was first applied to problems that involve a right triangle. This can be extended to non-right triangles (chapter 3), circles and circular motion, and a wide variety of applications.

As we shall see in the next unit, the six parts of any triangle (3 sides and 3 angles) are inherently linked through the processes of Trigonometry.

Given that the triangle is a **right triangle** we know:

- one angle is 90°;
- the side opposite the right angle is the longest in the triangle (and the smallest side is opposite the smallest angle);
- the remaining two angles are complementary, and so if we know one other angle we know all three angles in the triangle;
- the sides are related by Pythagoras' Theorem, and so if we know any two sides, we can find the third side of the triangle.

Using Trigonometry, if we are given a right triangle and the length of any two sides, we can determine the third side by using Pythagoras' Theorem. This in turn is sufficient information to calculate the trigonometric ratios of the angles of the triangle and consequently the measures of the angles.

18.2.1: THE RELATIONSHIP BETWEEN SIDES AND ANGLES OF A TRIANGLE

The sides of the triangle may also be named according to their relationship to a given angle.

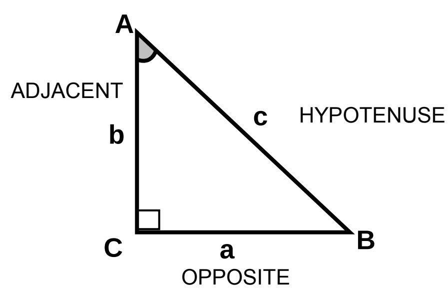
In a right triangle, the ***hypotenuse*** is always the side opposite the right angle.

The side opposite the angle is the called the ***opposite*** side; while the side which forms one arm of the angle is called the ***adjacent*** side.

Consider the triangle ΔABC shown.

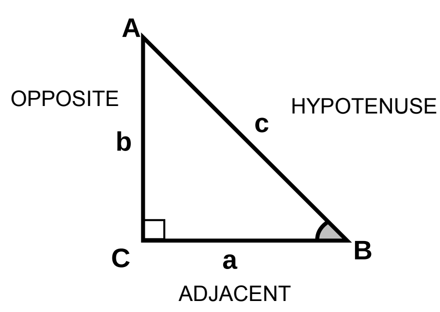
In relation to the angle ∠ A,

- o the side a is "opposite"
- o the side b is "adjacent".



In relation to the angle ∠ B,

- o the side a is "adjacent"
- o the side b is "opposite".



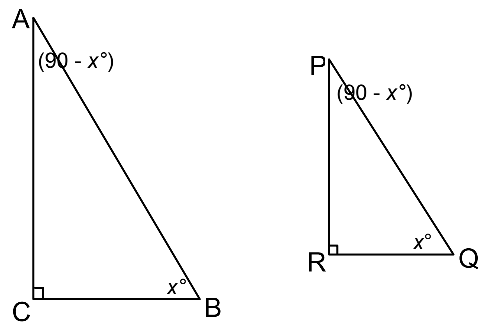
NOTE: These relationships do not apply to the right angle, ∠ C

Consider any two right triangles.

If the triangles have one other equal angle, x say, then the third angle of each triangle, (90° - x), must also be equal. Therefore the two triangles must be similar.

By section 1.3, since the two triangles are similar we know that pairs of corresponding sides are in fixed ratio.

$$\frac{AC}{BC} = \frac{PR}{QR} \qquad \frac{BC}{AB} = \frac{QR}{PQ} \qquad \frac{AC}{AB} = \frac{PR}{PQ}$$



Taking these ratios one at a time:

$$\frac{AC}{AB} = \frac{PR}{PQ} = \frac{\text{side opposite to } x^\circ}{\text{hypotenuse}}$$

$$\frac{BC}{AB} = \frac{QR}{PQ} = \frac{\text{side adjacent to } x^\circ}{\text{hypotenuse}}$$

$$\frac{AC}{BC} = \frac{PR}{QR} = \frac{\text{side opposite to } x^\circ}{\text{side adjacent to } x^\circ}$$

Notice also these ratios for can be related to the angle x° or to the complement of the angle, (90 - x°):

$$\frac{BC}{AB} = \frac{QR}{PQ} = \frac{\text{side opposite to } (90-x)^\circ}{\text{hypotenuse}} \quad \text{and}$$

$$\frac{AC}{AB} = \frac{PR}{PQ} = \frac{\text{side adjacent to } (90-x)^\circ}{\text{hypotenuse}}$$

To maintain consistency, note that $\frac{AC}{BC} = \frac{PR}{QR}$ gives us $\frac{BC}{AC} = \frac{QR}{PR}$ and so

$$\frac{BC}{AC} = \frac{QR}{PR} = \frac{\text{side opposite to } (90-x)^\circ}{\text{side adjacent to } (90-x)^\circ}$$

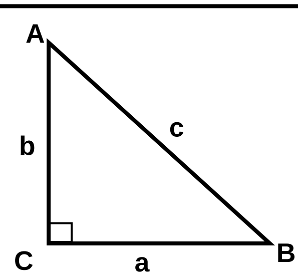
Because these fixed ratios are very important to us we assign them special names, as you will see in the next section.

18.2.2: THE DEFINITIONS OF THE TRIGONOMETRIC RATIOS IN A RIGHT TRIANGLE

$$\sin A = \frac{opposite}{hypotenuse} = \frac{a}{c}$$

$$\cos A = \frac{adjacent}{hypotenuse} = \frac{b}{c}$$

$$\tan A = \frac{opposite}{adjacent} = \frac{a}{b}$$



$$\sin B = \frac{opposite}{hypotenuse} = \frac{b}{c}$$

$$\cos B = \frac{adjacent}{hypotenuse} = \frac{a}{c}$$

$$\tan B = \frac{opposite}{adjacent} = \frac{b}{a}$$

NOTE: Like all ratios, the trigonometric ratios **do not have units**. Whatever units are used to measure the sides are cancelled out during the division process.

Trigonometric Ratios of Complimentary Angles

In a right triangle the two non-right angles must always sum to 90° and thus are always complimentary.

In the right triangle ΔABC, ∠C = 90°, the angles ∠A and ∠B are complimentary, and

$$\sin A = \cos B \qquad \text{and} \qquad \cos A = \sin B$$

The side b is adjacent to ∠ A and opposite to ∠ B; the side a is opposite to ∠ A and adjacent to ∠ B.

Thus:

$$\begin{aligned} \sin A &= \frac{\text{side opposite } \angle A}{\text{hypotenuse}} \\ &= \frac{a}{c} \\ &= \frac{\text{side adjacent } \angle B}{\text{hypotenuse}} \\ &= \cos B \end{aligned} \qquad \text{and} \qquad \begin{aligned} \cos A &= \frac{\text{side adjacent } \angle A}{\text{hypotenuse}} \\ &= \frac{b}{c} \\ &= \frac{\text{side opposite } \angle B}{\text{hypotenuse}} \\ &= \sin B \end{aligned}$$

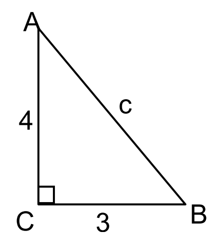
Finding the Trigonometric Ratios in a Right Triangle

Example 1: In ΔABC, ∠C = 90°, a = 3 and b = 4. Find:

a) the length of the hypotenuse, and

b) the values of sin A, cos A, tan A.

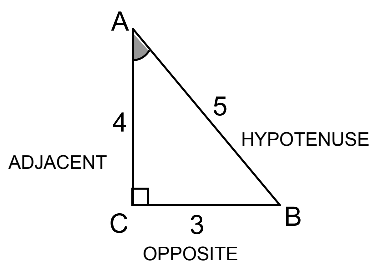
c) the values of sin B, cos B, tan B.



Solution: First use Pythagoras' Theorem to find the length of the hypotenuse c:

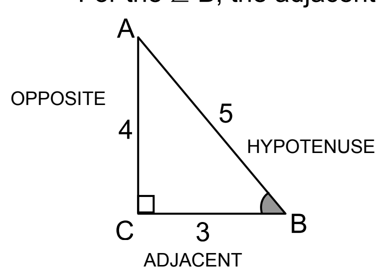
$$\begin{aligned} 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \\ c &= 5 \end{aligned}$$

For the ∠ A, the adjacent side measures 4 units and the opposite side measures 3 units.



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

For the ∠ B, the adjacent side measures 3 units and the opposite side measures 4 units.



$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

NOTE: $\sin A = \frac{3}{5} = \cos B$ and $\cos A = \frac{4}{5} = \sin B$

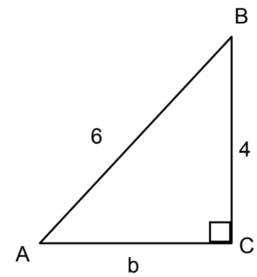
Example 2: In ΔABC, $\angle C = 90°$, $a = 4$ and $c = 6$. Find

(a) the length of the other leg (third side),and

(b) the values of sin A, cos A, tan A.

(c) the values of sin B, cos B, tan B.

Solution: First, draw a diagram labeling the data given and the unknown values.

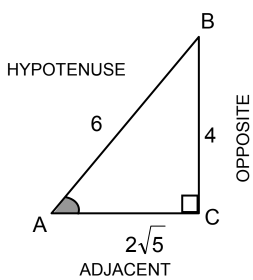


Use Pythagoras' Theorem to find the length of the third side of the triangle, b.

$$\begin{aligned} 4^2 + b^2 &= 6^2 \\ 16 + b^2 &= 36 \\ b^2 &= 20 \\ b &= \sqrt{20} \\ b &= 2\sqrt{5} \end{aligned}$$

$$NOTE: \quad \sqrt{20} = \sqrt{4}\cdot\sqrt{5} = 2\sqrt{5}$$

For the $\angle$ A, the adjacent side measures $2\sqrt{5}$ and the opposite side measures 4.



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{6} = \frac{2}{3}$$

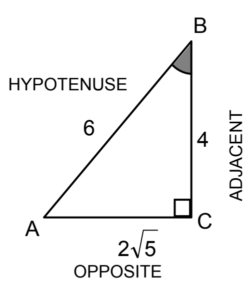
$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$$

rationalize the denominator:

$$\tan A = \frac{2}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

For the $\angle$ B, the adjacent side measures 4 and the opposite side measures $2\sqrt{5}$.



$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$$

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{6} = \frac{2}{3}$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{2\sqrt{5}}{4} = \frac{\sqrt{5}}{2}$$

NOTE: $\sin A = \frac{2}{3} = \cos B$ and $\cos A = \frac{\sqrt{5}}{3} = \sin B$

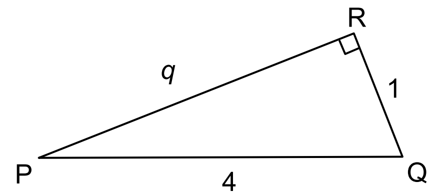
Example 3: In ΔPQR, $\angle R = 90°$, $p = 1$ and $r = 4$. Find the length of the other leg (third side),of the triangle and:

(a) cos Q

(b) tan P

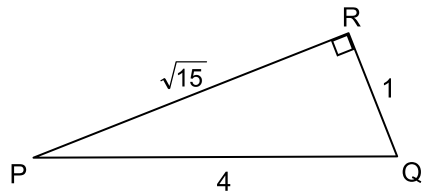
Solution: Draw a diagram labeling the data given and the unknown values.

Using Pythagoras' Theorem to find q



$$\begin{aligned} 1^2 + q^2 &= 4^2 \\ 1 + q^2 &= 16 \\ q^2 &= 15 \\ q &= \sqrt{15} \end{aligned}$$

For $\angle$ Q, the adjacent side measures 1 unit; for $\angle$ P the adjacent side measures $\sqrt{15}$ units and the opposite side measures 1 unit.



$$\cos Q = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{4}$$

$$\tan P = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{15}} = \frac{1}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \frac{\sqrt{15}}{15}$$

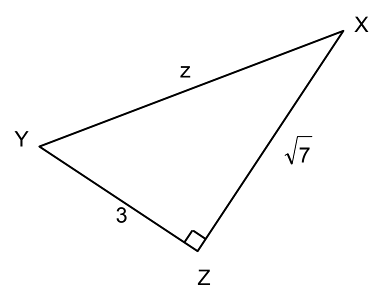
Example 4: In ΔXYZ, $\angle Z = 90°$, $x = 3$ and $y = \sqrt{7}$. Find

(a) the length of the hypotenuse z

(b) tan Y

(c) sin X

Solution: Draw a diagram labeling the data given and the unknown values to be found.



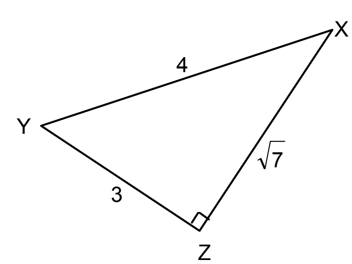
Using Pythagoras' Theorem to find z:

$$\begin{aligned} 3^2 + \left(\sqrt{7}\right)^2 &= z^2 \\ 9 + 7 &= z^2 \\ z^2 &= 16 \\ z &= 4 \end{aligned}$$

Substituting this value for z we can now find the trigonometric ratios:

$$\tan Y = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{7}}{3}$$

$$\sin X = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{4}$$



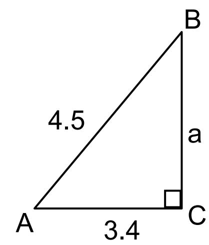
Example 5: Using the calculator to approximate values

In ΔABC, ∠C is a right angle, b = 3.4 and c = 4.5. Find

(a) the length of the third side of the triangle, correct to one decimal place.

(b) sin A, cos A, and tan A, approximated correct to the nearest thousandth.

(c) sin B, cos B, and tan B, approximated correct to the nearest thousandth.



Solution: Using Pythagoras' Theorem to find a:

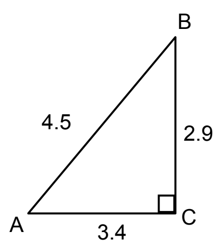
$$a^2 + b^2 = c^2$$
$$a^2 + 3.4^2 = 4.5^2$$
$$a^2 = 4.5^2 - 3.4^2$$
$$a = \sqrt{4.5^2 - 3.4^2}$$

Calculator entry:

$\boxed{\sqrt{\ }}$ $\boxed{(}$ **4.5** $\boxed{x^2}$ $\boxed{-}$ **3.4** $\boxed{x^2}$ $\boxed{)}$ $\boxed{=}$

$$a = 2.94788$$
$a \approx 2.9$ correct to one decimal place

Substituting this value for c in ΔABC we can find the six trigonometric ratios:



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{2.9}{4.5} \approx 0.644$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3.4}{4.5} = 0.756$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{2.9}{3.4} \approx 0.853$$

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3.4}{4.5} \approx 0.756$$

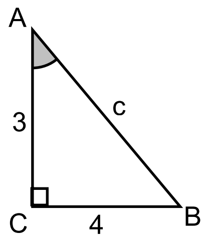
$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2.9}{4.5} = 0.644$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{3.4}{2.9} \approx 1.172$$

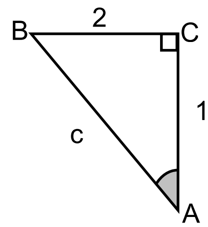
Exercise 2.2

In each right triangle ΔABC described below, C = 90°. Find the exact values of sin A, cos A and tan A.

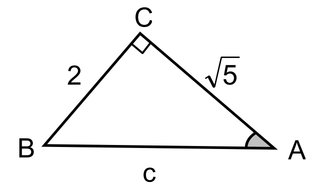
1.



2.

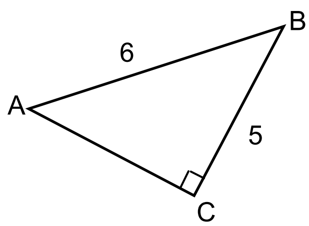


3.

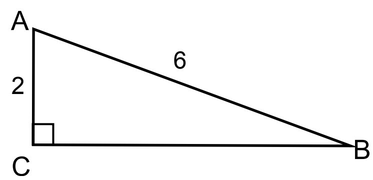


In each right triangle below, find the exact value of sin A, cos A, tan A, and sin B, cos B, and tan B.

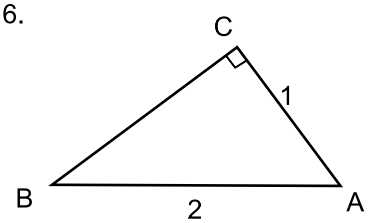
4.



5.



6.



7. In ΔABC, ∠C = 90°, a = 40, and b = 30. Find the exact values of tan A and cos B

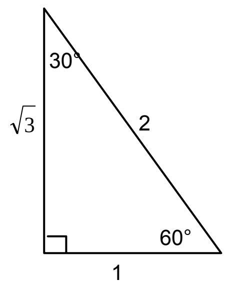
8. In ΔPQR, ∠R = 90°, p = 5, and r = 13. Find the exact values of cos P and tan Q

9. In ΔDEF, ∠E = 90°, d = 4.8, and e = 6.5. Find the value of cos D correct to three decimal places.

10. In ΔXYZ, ∠X = 90°, x = 12.8, and y = 7.5. Find the value of tan Z correct to three decimal places.

18.2.3: EXACT TRIGONOMETRIC RATIOS FOR 30°, 45° AND 60° ANGLES.

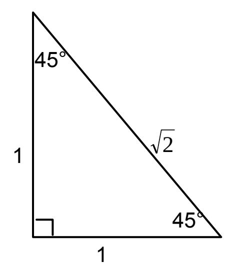
Exact Trigonometric Ratios for 30° and 60°.



Since the sides of any 30° - 60° - 90° triangle are in the fixed ratio of $1 : \sqrt{3} : 2$ we may use this triangle to determine the exact trigonometric ratios for 30° and 60°. From this triangle we see that:

$$\sin 30^\circ = \frac{1}{2} \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}$$
$$\cos 30^\circ = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60^\circ = \frac{1}{2}$$
$$\tan 30^\circ = \frac{1}{\sqrt{3}} \qquad \tan 60^\circ = \sqrt{3}$$

Exact Trigonometric Ratios for 45°

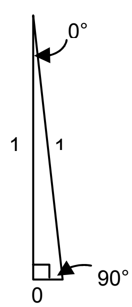


The sides of any isosceles right triangle are in the fixed ratio of $1 : 1 : \sqrt{2}$.

From this triangle we see that

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = 1$$

Exact Trigonometric Ratios for 0° and 90°



With a little bit of imagination we can conceptually derive the exact ratios for 0° and 90°.
Consider a right triangle, with another angle of 90°. The third angle must be 0°. If this triangle has one side of 1 unit, the other side must be 0 units in length, and the hypotenuse will be 1 unit.
From this triangle we see that:

$$\sin 0^\circ = \frac{0}{1} = 0 \qquad \sin 90^\circ = \frac{1}{1} = 1$$
$$\cos 0^\circ = \frac{1}{1} = 1 \quad \text{and} \quad \cos 90^\circ = \frac{0}{1} = 0$$
$$\tan 0^\circ = \frac{0}{1} = 0 \qquad \tan 90^\circ = \frac{1}{0} = \textit{undefined}$$

Summary:

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

18.2.4: USING THE CALCULATOR TO FIND TRIGONOMETRIC RATIOS

Most calculators work in one of two ways when computing the values for the trigonometric functions. You need to be familiar with yours.

First check that the MODE of the calculator is degrees [DEG] and not radians [RAD]

CALCULATOR TYPE 1

1. Press $\boxed{\text{SIN}}$, $\boxed{\text{COS}}$ or $\boxed{\text{TAN}}$
2. Enter a number in decimal degrees
3. Press equal/enter to get the value

CALCULATOR TYPE 2 (older style)

1. Enter a number in decimal degrees
2. Press $\boxed{\text{SIN}}$, $\boxed{\text{COS}}$ or $\boxed{\text{TAN}}$

Note: the equals sign is not needed here.

Round the value to the required number of decimal places.

Example 1: Find the value of, approximated correct to the nearest thousandth:

(a) sin 75° (b) cos 42° (c) tan 65.9°

(a) Find sin 75° to the nearest thousandth

Enter: ***Calculator Type 1***

$\boxed{\text{sin}}$ 75 $\boxed{=}$

Calculator Type 2

75 $\boxed{\text{sin}}$

Display: the number of decimal places given in the calculator's answer will depend on your calculator:

.965925826

Rounding: we want the "nearest thousandth" we approximate correct to three decimal places:

.965|925826

and since the digit in the fourth decimal place is 9, we round the 5 in the third place up to 6.

Answer: sin 75° $\approx$ 0.966

(b) Find cos 42° to the nearest thousandth

Enter: ***Calculator Type 1***

$\boxed{\text{cos}}$ 42 $\boxed{=}$

Calculator Type 2

42 $\boxed{\text{cos}}$

Display:

.7431448254

Rounding: approximate correct to three decimal places

.743 | 1448254

since the digit in the fourth decimal place is 1, leave the number in the third place as 3.

Answer: cos 42° ≈ 0.743

(c) Find tan 65.9° to the nearest thousandth

Enter: ***Calculator Type 1***

tan 65.9 =

Calculator Type 2

65.9 tan

Display: 2.2355279995

Rounding: approximate correct to three decimal places

2. 235 | 5279995

since the digit in the fourth decimal place is 5, round the 5 in the third place up to 6.

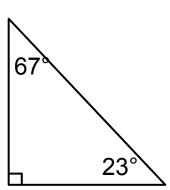
Answer: tan 65° ≈ 2.236

Example 2:

Draw a diagram of a right triangle which has an angle of 23°. Find

a) the size of the third angle, and

b) the sine, cosine and tangent ratios of each angle correct to the nearest thousandth.



a) Since the non-right angles of a right triangle are complementary (i.e. add to 90°) the third angle is:

90° - 23° = 67°

(Or, since the angle sum of a triangle is 180°, the third angle is:
180° – 90° - 23° = 67°)

b) Using a calculator:

$sin\ 23^0 = 0.391$ $sin\ 67^0 = 0.921$

$cos\ 23^0 = 0.921$ $cos\ 67^0 = 0.391$

$tan\ 23^0 = 0.424$ $tan\ 67^0 = 2.356$

Exercise 18.2.4

Questions 1 – 10: Use a calculator to find each of the following. Approximate all answers correct to the nearest thousandth (three decimal places.)

1) sin 42°

2) tan 13°

3) tan 87.3°

4) cos 75°

5) cos 48.8°

6) sin 8°

7) cos 12.9°

8) sin 22.5°

9) tan 29.78°

10) tan 85°

Questions 11 - 14: Use a calculator to find the value of each of following. Approximate all answers correct to the nearest thousandth (three decimal places.)
NOTE: In each case, the trigonometric functions are **cofunctions** of one another, and the angles are complementary angles.

11) sin13°, cos 77°

12) sin 5°, cos 85°

13) sin 0°, cos 90°

14) sin 90°, cos 0°

Questions 15 - 17: Draw a diagram of a right triangle which has the given angle. Find

a) the size of the third angle, and
b) the sine, cosine and tangent ratios of each angle correct to the nearest thousandth.

15) 30°

16) 56.7°

17) 45°

Questions 18 – 20: Complete the following table using the values computed and then use these values to answer the questions below by completing the sentence.

	0°	13°	23°	30°	42°	45°	56.7°	60°	67°	75°	85°	90°
sin			0.391						0.921	0.966	0.996	
cos		0.974	0.921		0.743				0.391			
tan	0		0.424		0.900				2.356			undefined

18) As the acute angle gets larger, the sine of the angle gets _________ and the cosine of the angle gets _________

19) The value of the sine and cosine of an angle always lies between the numbers _____ and _____

20) The smallest value of the tangent of an angle is _________ and the biggest value is _________

[HINT: In the last question, use your calculator to look at the values of tan 89°, tan 89.9°, tan 89.99°, tan 89.999°, tan 89.9999°, etc]

18.2.5: USING TRIGONOMETRIC RATIOS TO FIND AN ANGLE

Most calculators work in one of two ways when computing the angle given the value of the trigonometric ratio. You need to be familiar with yours.

Again, first check that the MODE of the calculator is degrees [DEG] and not radians [RAD]

NOTATION: If $\sin A = x$ then $A = \sin^{-1} x$

If $\cos A = x$ then $A = \cos^{-1} x$

If $\tan A = x$ then $A = \tan^{-1} x$

CALCULATOR TYPE 1

1. Press INV or 2^ND button
2. Press SIN, COS or TAN
3. Enter the value of the ratio
4. Press equal/enter to get the value

Round the resulting angle to the required number of decimal places.

CALCULATOR TYPE 2 (older style)

1. Enter the value of the ratio
2. Press INV or 2^ND button
3. Press SIN, COS or TAN

Note: the equals sign is not needed.

Example 1: Find the angle, correct to the nearest tenth of a degree, given the ratio.

(a) sin A = 0.456 (b) cos B = 0.122 (c) tan θ = 3.245

(a) If sin A = 0.456, find the angle A correct to the nearest tenth of a degree, given the ratio.

Enter: ***Calculator Type 1***

Use the INV or 2^ND button

Calculator Type 2

Display:

Type 1: INV sin 0.456 =

Type 2: 0.456 INV sin

the number of decimal places given in the calculator's answer will depend on your calculator:

27.12929446

Rounding: we want the "nearest tenth of a degree" we approximate correct to one decimal place:

27.1|2929446

and since the digit in the second decimal place is 2, which is less than 5, we leave the number in the first decimal place as 1.

Answer: If sin A = 0.456 then A = 27.1°

Check: Using the calculator, sin 27.1° = 0.4555449… ≈ 0.456

(b) If cos B = 0.122, find the angle A correct to the nearest tenth of a degree, given the ratio.

Enter: ***Calculator Type 1*** | ***Calculator Type 2***

Use the [INV] or [2^{ND}] button

Display: [INV] [cos] 0.122 [=] | 0.122 [INV] [cos]

the number of decimal places given in the calculator's answer will depend on your calculator:

82.99245764

Rounding: we want the "nearest tenth of a degree" we approximate correct to one decimal place:

82.9|9245764

Since the digit in the second decimal place is 9, which is more than 5, we round the 9 in the first decimal place to 10, which will round 82.9 to 83.0.

Answer: If cos B = 0.122 then B = 83.0°

NOTE: To correctly answer this question, we should write 82.0° not 82 in order to indicate that this angle is correct to the nearest tenth of a degree.

Check: Using the calculator, cos 83° = 0.1218693… ≈ 0.122

(c) If tan θ = 3.245, find the angle θ correct to the nearest tenth of a degree, given the ratio.

NOTE: *θ is the greek letter "theta", commonly used in trigonometry to represent an angle.*

Enter: ***Calculator Type 1*** | ***Calculator Type 2***

Use the [INV] or [2^{ND}] button

Display: [INV] [tan] 3.245 [=] | 3.245 [INV] [tan]

72.8|7245960

Rounding: Round the 8 in the first decimal place to 9

Answer: If tan θ = 3.245 then θ = 72.9°

Check: Using the calculator, tan 72.9° = 3.2505500… ≈ 3.251

This is not 3.245 as we would like. However, considering the alternatives,

tan 73.0° = 3.27085… ≈ 3.271 ,

tan 72.9° = 3.25055… ≈ 3.251 and

tan 72.8° = 3.23047… ≈ 3.230,

so we have the closest approximation to one tenth of a degree.

Exercise 18.2.5

Questions 1 – 8: Use a calculator to find each of the following. Approximate all answers as instructed.

1) A, to the nearest degree, if $\sin A = 0.200$

2) θ, to the nearest tenth of a degree if, $\tan\theta = 3.45$

3) P, to the nearest degree if $\cos P = 0.1022$

4) B if $\cos B = \frac{\sqrt{3}}{2}$

5) θ, to the nearest degree, if $\cos\theta = 0.893$

6) A, if $\sin A = \frac{\sqrt{2}}{2}$

7) θ, to the nearest tenth of a degree, if $\tan\theta = 0.093$

8) B if $\tan B = \frac{\sqrt{3}}{3}$

18.2.6: SOLVING RIGHT TRIANGLES

The three sides and three angles of any triangle are related through the trigonometric ratios, and we are able to use trigonometry to determine the measure of the remaining components if we are given any three facts about the triangle.

In this section we are solving right triangles. That is, we know that the triangle has one angle that of 90°. Given this and:

- the length of two sides, or
- the size of one angle and the length of one side,

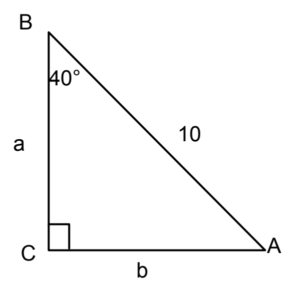
we can find the remaining sides and angles of the triangle.

In Chapter 3 we will extend our theory to non-right triangles.

Solve the Right Triangle Given One Angle and One Side

Example 1: Find the remaining sides and angles of the triangle ΔABC, given that ∠C is a right angle, ∠B = 40° and c = 10 cm.

Solution: <u>The first step is always to draw a diagram of the triangle.</u> Label the diagram with the information given and the unknowns to be found.



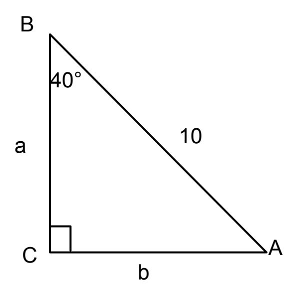
In this example we know ∠B, ∠C and c.

We need to find ∠A, a, and b

<u>To find ∠A</u> we can use the fact that the two acute angles of the triangle are complementary,

$$\begin{aligned} \angle A &= 90^\circ - \angle B \\ &= 90^\circ - 40^\circ \\ \therefore \quad \angle A &= 50^\circ \end{aligned}$$

To find a and b we need to identify their relative positions in a trigonometric ratio of which we know two out of the three components.



<u>To find a</u>, we can use either cos B (since *a* is the adjacent side to the given angle ∠B, and we know the hypotenuse) or sin A (since *a* is the opposite side to the calculated angle ∠A, and we know the hypotenuse).

Using cos B: $cos\, B = \dfrac{adjacent}{hypotenuse}$

Substituting the known values for ∠B and c,

$$cos\, 40^\circ = \frac{a}{10}$$

Multiplying both sides of the equation by 10:

$$a = 10\, cos\, 40^\circ$$

Calculator:
10 $\boxed{\text{x}}$ $\boxed{\text{cos}}$ 40 $\boxed{=}$

Using a calculator

$$a \approx 7.6604$$

Since the least number of significant figures in the data is three, answers in this problem should be rounded correct to three significant figures: $a \approx 7.7$

ALTERNATE: Find a using sin A:

$$\sin A = \frac{opposite}{hypotenuse}$$

Substituting the known values of ∠B and c :

$$\sin 50° = \frac{a}{10}$$

Multiplying both sides by 10 :

$$a = 10\sin 50°$$

Using a calculator :

$$a \approx 7.7$$

Calculator:

10 $\boxed{x}$ $\boxed{\sin}$ 50 $\boxed{=}$

<u>To find b</u>, we can use either sin B (since b is the opposite side to the given angle ∠B, and we know the hypotenuse) or cos A (since b is the adjacent side to the given angle ∠B, and we know the hypotenuse), or we could use Pythagoras' Theorem.

Using sin B:

$$\sin B = \frac{opposite}{hypotenuse}$$

Substituting the known values of ∠B and c :

$$\sin 40° = \frac{b}{10}$$

Multiplying both sides by 10 :

$$b = 10\sin 40°$$

Using a calculator :

$$a \approx 6.4$$

Calculator:

10 $\boxed{x}$ $\boxed{\sin}$ 40 $\boxed{=}$

ALTERNATE: Find b using cos A:

$$\cos A = \frac{adjacent}{hptotenuse}$$

Substituting the known values of ∠B and c :

$$\cos 50° = \frac{b}{10}$$

Multiplying both sides by 10 :

$$b = 10\cos 50°$$

Using a calculator :

$$a \approx 6.4$$

Calculator:

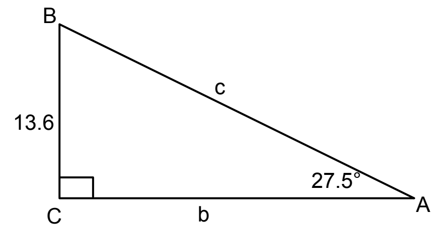
10 $\boxed{x}$ $\boxed{\cos}$ 50 $\boxed{=}$

ALTERNATE: We could also find b using Pythagoras' Theorem. This is the least appealing method because it uses an approximated calculation rather than an exact measure.

NOTE: While checking our calculations, notice that the smallest side, b = 6.4 cm, is opposite the smallest angle, ∠B = 40°; the "middle" side, a = 7.7 cm, is opposite the middle angle, ∠A = 50°, and the hypotenuse, c = 10 cm, is the longest side of the triangle.

Example 2: Solve the triangle ΔABC, given ∠C is a right angle, ∠A = 27.5° and a = 13.6 units.

Solution: To find ∠B:



$$\begin{aligned}\angle B &= 90° - \angle A \\ &= 90° - 27.5° \\ &= 62.5°\end{aligned}$$

To find b tan*B* is chosen instead of tan*A* only because it results in an slightly easier manipulation:

$$\tan B = \frac{opposite}{adjacent}$$

$$\tan 62.5° = \frac{b}{13.6}$$

$$\begin{aligned}b &= 13.6 \tan 62.5° \\ &\approx 26.1\end{aligned}$$

NOTE: Since the least number of significant figures in the data is three, answers in this problem should be rounded correct to three significant figures.

To find c:

$$\sin A = \frac{opposite}{hypotenuse}$$

$$\therefore \quad c = \frac{a}{\sin A}$$

$$c = \frac{13.6}{\sin 27.5°}$$

$$\approx 29.5$$

Calculator:
13.6 $\boxed{\div}$ $\boxed{\text{sin}}$ 27.5 $\boxed{=}$

Example 3: Find the exact values of the remaining sides and angles of ΔABC, given ∠C is a right angle, ∠A = 45° and a = 3 units.

Solution: Since one angle of ΔABC is 45° we know that
- the triangle is an isosceles triangle with two angles of 45°;
- the sides opposite these angles are equal, and both 3 units

Therefore ∠B = 45° and b = 3.

In fact we also know that the sides are in the fixed ratio of 1:1: $\sqrt{2}$ and do not need to use trigonometric ratios to find c in this case.

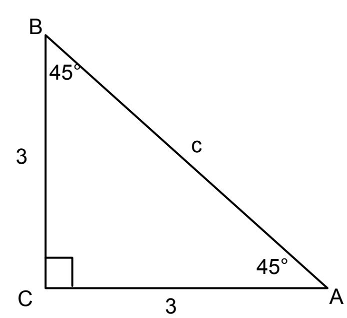
However to demonstrate the use of trigonometry, we will use the sine ratio to find c.

To find c:

$$\sin 45° = \frac{opposite}{hypotenuse}$$

$$\sin 45° = \frac{3}{c}$$

$$\therefore \quad c = \frac{3}{\sin 45°}$$



We know $\sin 45° = \frac{1}{\sqrt{2}}$, hence

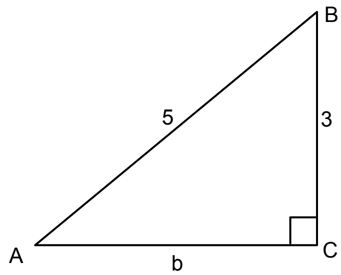
$$c = \frac{3}{1/\sqrt{2}}$$

$$c = 3\sqrt{2}$$

Solve the Right Triangle Given the Length of Two Sides

Example 4: Solve the triangle ΔABC, given $\angle C$ is a right angle, a = 3 cm and c = 5 cm.

Solution: Draw a diagram and label all known and unknown values.



To find $\angle A$

$$\sin A = \frac{opposite}{hypotenuse}$$

$$\sin A = \frac{3}{5}$$

$$A = \sin^{-1}\left(\frac{3}{5}\right)$$

$$\approx 37°$$

Calculator:

$\boxed{\text{INV}}$ $\boxed{\sin}$ $\boxed{(}$ 3 $\boxed{\div}$ 5 $\boxed{)}$ $\boxed{=}$

The easiest method to find $\angle B$ is to use the fact that it is the complement of $\angle A$.

$$\begin{aligned}\angle B &= 90° - \angle A \\ &= 90° - 37° \\ &= 53°\end{aligned}$$

Alternately, we can use the cosine ratio:

$$\cos B = \frac{adjacent}{hypotenuse}$$

$$\cos B = \frac{3}{5}$$

$$B = \cos^{-1}\left(\frac{3}{5}\right)$$

$$\approx 53°$$

Calculator:

$\boxed{\text{INV}}$ $\boxed{\cos}$ $\boxed{(}$ 3 $\boxed{\div}$ 5 $\boxed{)}$ $\boxed{=}$

The easiest method to find b is to use Pythgoras' Theorem – or better yet, to recognize the missing element of the {3, 4, 5} Pythagorean Triad – to find that the exact measure of b is 4 units.

Alternately, we can use the sine, cosine or tangent ratios to find b:

$$\tan A = \frac{opposite}{adjacent}$$

$$\tan 37° = \frac{3}{b}$$

Rearranging :

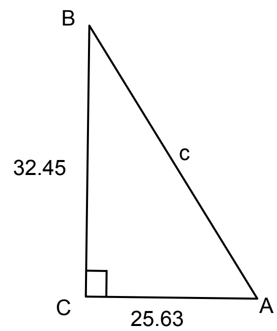
$$b = \frac{3}{\tan 37°}$$

$$\approx 4$$

This is the least appealing method because we are using an approximated calculation when the more accurate given data could be used.

Example 5: Solve the triangle ΔABC, given ∠C is a right angle, a = 32.45 cm and b = 25.63 cm.

Solution: Since we are given the opposite and adjacent sides for ∠A, we can use the tan ratio to find the angle.



$$\tan A = \frac{opposite}{adjacent}$$

$$\tan A = \frac{32.45}{25.63}$$

$$A = \tan^{-1}\left(\frac{32.45}{25.63}\right)$$

$$\angle A \approx 51.70^\circ$$

Since ∠A and ∠B are complementary:

$$\begin{aligned}\angle B &= 90^\circ - \angle A \\ &= 90^\circ - 51.70^\circ \\ \angle B &\approx 38.30^\circ\end{aligned}$$

Using Pythagoras' Theorem, we can find c.

$$\begin{aligned}32.45^2 + 25.63^2 &= c^2 \\ c &= \sqrt{32.45^2 + 25.63^2} \\ c &\approx 41.35\end{aligned}$$

> We can use the tangent ratio to find ∠B:
>
> $$\tan B = \frac{opposite}{adjacent}$$
>
> $$\tan A = \frac{25.63}{32.45}$$
>
> $$A = \tan^{-1}\left(\frac{25.63}{32.45}\right)$$
>
> $$\angle A \approx 38.30$$

Solve the Right Triangle Given a Trigonometric Ratio

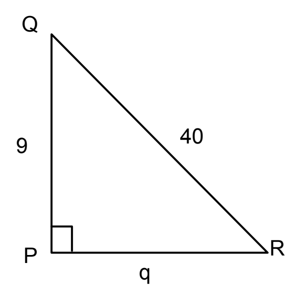
Example 6: Find the sides and angles of a triangle ΔPQR, where ∠P is a right angle and $\sin R = \frac{9}{40}$.

Solution: Since $\sin R = \frac{opposite}{hypotenuse}$ and $\sin R = \frac{9}{40}$ then we can use

$$\sin R = \frac{opposite}{hypotenuse} = \frac{9}{40}$$

That is, since we know that the ratio of the sides is 9 to 40, we can set the side opposite ∠R as 9 units and the hypotenuse of the triangle as 40 units.

Draw a diagram and label vertices and sides.



By Pythagoras' Theorem,

$$\begin{aligned}9^2 + q^2 &= 40^2 \\ q &= \sqrt{40^2 - 9^2} \\ &= \sqrt{1519} \\ q &\approx 39\end{aligned}$$

Since $\sin R = \frac{9}{40}$ then,

$$R = \sin^{-1}\left(\frac{9}{40}\right)$$

$$\angle R \approx 13^\circ$$

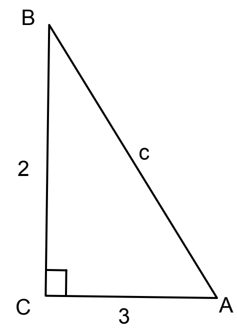
∠Q and ∠R are complementary so:

$$\angle Q = 90^\circ - 13^\circ$$

$$\angle Q = 77^\circ$$

Example 7: In right triangle ΔABC, where ∠C is a right angle, find cos B given that $\tan A = \frac{2}{3}$.

Solution: Since $\tan A = \frac{opposite}{adjacent}$ and $\tan A = \frac{2}{3}$ then for ∠A in ΔABC



$$\frac{opposite}{adjacent} = \frac{2}{3}$$

At this point, a diagram is very useful!

We need to find $\cos B$.

$$\cos B = \frac{adjacent}{hypotenuse}$$
$$= \frac{3}{c}$$

Using Pythagoras' Theorem to find c:

$$2^2 + 3^2 = c^2$$
$$c = \sqrt{2^2 + 3^2}$$
$$c = \sqrt{13}$$

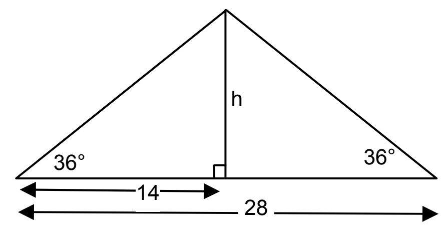
Therefore $\cos B = \frac{3}{\sqrt{13}}$

Applications

Example 9: Find the altitude of an isosceles triangle which has a base of length 28 cm and base angles measuring 36°.

Solution: We need to find h.

Since h is the side opposite the angle 36° and we have the measure of the side adjacent to 36° we will use the tangent ratio.



$$tan 36° = \frac{h}{14}$$

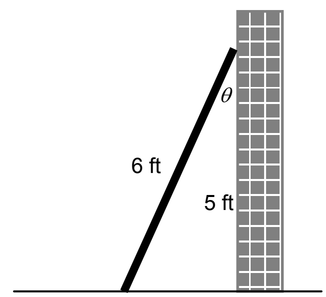
$$\therefore \quad h = 14\, tan 36°$$

$$\approx 10.17$$

Since we need to approximate correct to 2 significant figures, the height of the triangle is calculated as 10 cm.

Example 10: A 6 ft ladder leans against a wall and reaches to a height of 5 ft. What angle does the ladder form with the wall?

Solution: Draw a diagram. Let θ be the angle



$$cos\,\theta = \frac{5}{6}$$

$$\therefore \quad \theta = cos^{-1}\left(\frac{5}{6}\right)$$

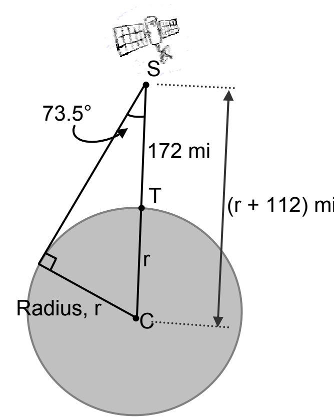
$$\theta \approx 34°$$

Therefore the ladder makes a 34° angle with the wall.

Example 11: A satellite is orbiting 172 miles above the earth's surface. (See diagram) When it is directly above the point T, the angle S is found to be 73.5°. Find the radius of the earth.

Solution: Let the radius of the earth be r miles.

Then:



$$\sin 73.5° = \frac{r}{r+172}$$

$$(r+172)\sin 73.5° = r$$

$$r \sin 73.5° + 172 \sin 73.5° = r$$

$$172 \sin 73.5° = r - r \sin 73.5°$$

$$= r(1 - \sin 73.5°)$$

$$r = \frac{172 \sin 73.5°}{1 - \sin 73.5°}$$

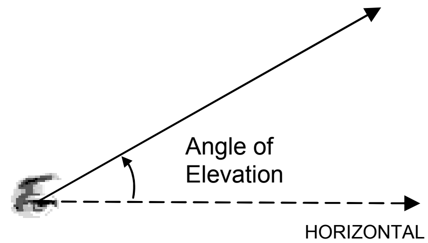
$$r \approx 4004.8$$

Hence the radius of the earth is approximately 4000 miles (correct to three significant figures.)

Angles of Elevation and Depression

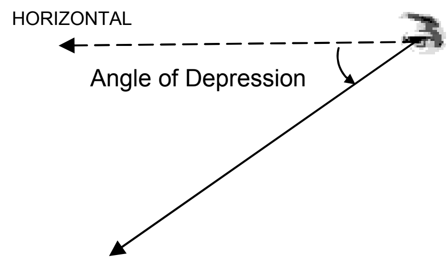
Angles of Elevation and Depression are measured relative to the observer.

An imaginary line drawn from the eye of the observer to the object being observed is called the **line of sight.**



The **horizontal** is the line of sight to an object directly in front of, neither above nor below, the eye of the observer.

If the object being observed is above the horizontal, then the angle between the line of sight and the horizontal is called the **angle of elevation**. (Another way to think of this is the angle the observer would need to *look up* to the object.)



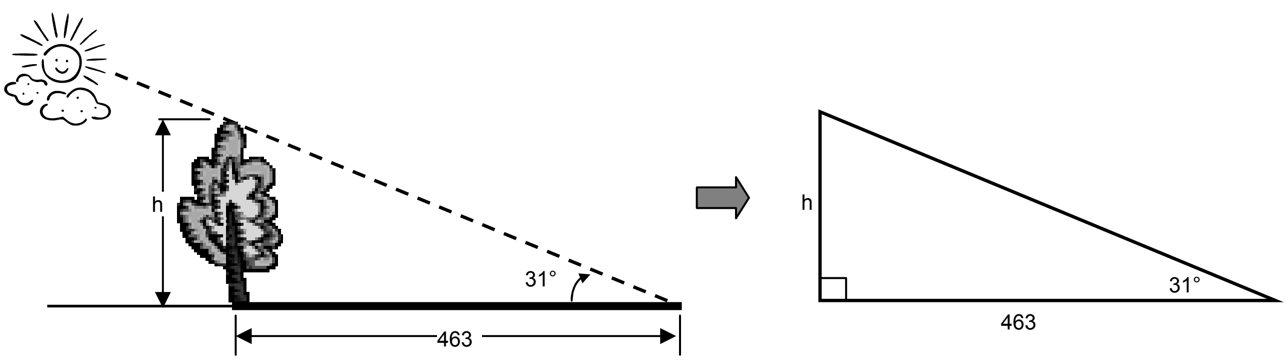
If the object is below the horizontal, then the angle between the line of sight and the horizontal is called the **angle of depression**. (Another way to think of this is the angle the observer would need to *look down* to the object.)

Note that the angle of elevation or depression is always measured from the observer to the object.

Example 12: To determine the height of a tree, a student observes that it casts a 463 ft shadow when the angle of elevation of the sun (from the top of the shadow) is 31°. Find the height of the tree.

Solution: We need to find h, the height of the tree.

First draw a diagram.



The top of the tree, base of the tree/foot of the shadow, and tip of the shadow form the three vertices of a right triangle.

We know one angle and two sides of the triangle: the angle of elevation, the length of the shadow which is an **adjacent** side of the angle; and want to find the height of the tree, which is the **opposite** side of the triangle to the angle. Thus we use the tangent ratio.

$$\tan 31^\circ = \frac{h}{463}$$

$$\therefore \quad h = 463 \tan 31^\circ$$

$$h \approx 278$$

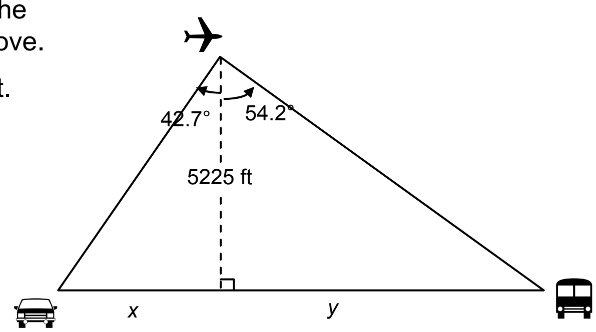
The tree is approximately 278 ft in height.

Example 13: An airplane is flying at an altitude of 5225 ft, directly above a straight stretch of highway along which a car and a bus are traveling towards each other. The vehicles are on opposite sides of the airplane, the car at an angle of depression of 42.7° and the bus at an angle of depression of 54.2° from the plane. How far apart are the vehicles to the nearest tenth of a mile?

Solution: Let the car be x ft and the bus be y ft from the point which the airplane is flying directly above.

Then the car and the bus are (x + y) ft apart.

Since we have the measure of the angle and its adjacent side, and we wish to find the opposite side, we use the tangent ratio.



$$\tan 42.7^\circ = \frac{x}{5225}$$
$$x = 5225 \tan 42.7^\circ$$

$$\tan 54.2^\circ = \frac{y}{5225}$$
$$y = 5225 \tan 54.2^\circ$$

$$\begin{aligned} x + y &= 5225 \tan 42.7^\circ + 5225 \tan 54.2^\circ \\ &= 5225(\tan 42.7^\circ + \tan 54.2^\circ) \\ &\approx 12066\ ft \\ &= \frac{12066}{5280}\ miles \\ &\approx 2.285\ miles \end{aligned}$$

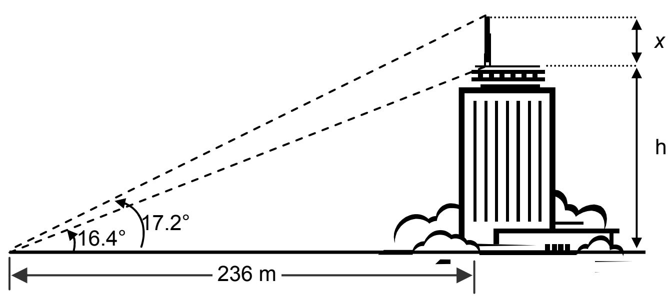
Hence the vehicles are approximately 2.3 miles apart.

Example 14: From a point 236.0 meters from the base of a building, the angle of elevation to the top of the building is 16.4°. The angle of elevation from the same point to the tip of a flagpole on top of this building is 18.5°. What is the height of the flagpole?

Solution: Let the building be h meters and the flagpole be x meters in height.

Then:

$$\tan 16.4^\circ = \frac{h}{236}$$
$$h = 236 \tan 16.4^\circ$$



and

$$\tan 17.2^\circ = \frac{x+h}{236}$$
$$x + h = 236 \tan 18.5^\circ$$
$$x = 236 \tan 18.5^\circ - h$$

Substituting for h:

$$\begin{aligned} x &= 236 \tan 18.5^\circ - h \\ &= 236 \tan 18.5^\circ - 236 \tan 16.4^\circ \\ &= 236(\tan 18.5^\circ - 236 \tan 16.4^\circ) \\ &\approx 9.5059 \end{aligned}$$

Hence the pole is approximately 9.5 m tall

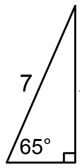
Exercise 18.2.6

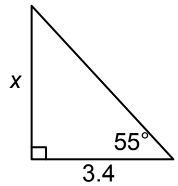
1. Solve the triangle ΔABC, given that ∠C is a right angle, and:

   (a) a = 10 and b = 24 (b) ∠A = 48° and c = 12 (c) a = 1.6 and c = 2.4

   (d) ∠B = 34° and b = 17 (e) ∠A = 18.6° and b = 18.8 (f) b = 56 and c = 124

2. Find the value of the side labeled *x* correct to the nearest tenth.

   (a)


   (b)


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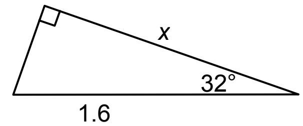

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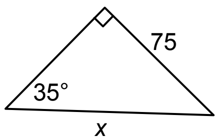

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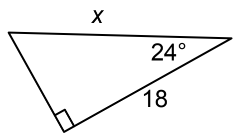

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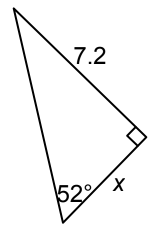
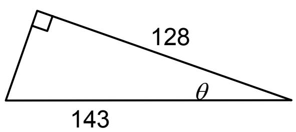


3. Find the angle θ correct to the nearest degree.

   (a)


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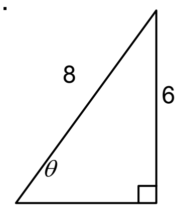

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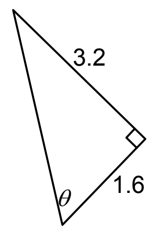

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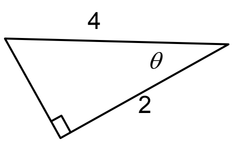

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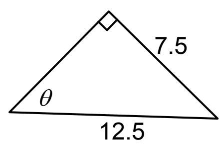

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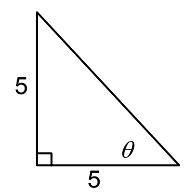
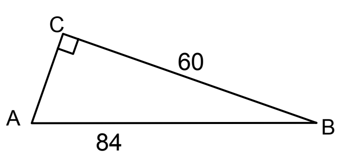


4. Solve the right triangle

   (a)


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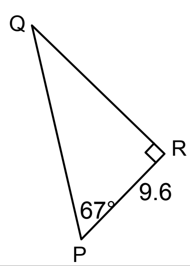

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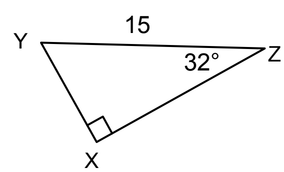
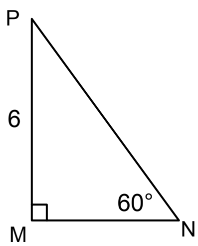


5. Solve for the exact values of the right triangle

   (a)


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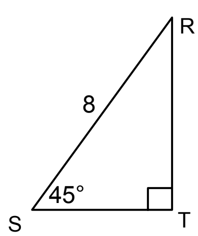

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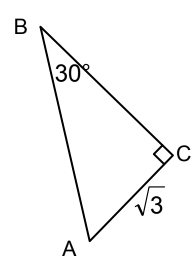


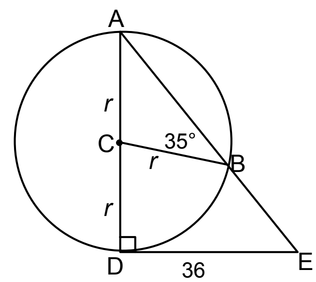
6. A ladder 10 ft in length reaches 9 ft up a wall against which it leans. Find the angle, to the nearest degree, that the ladder makes with the wall.

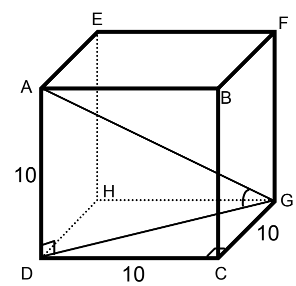
7. A road up a hill makes an angle of 4.5° with the horizontal. If this road is 5.2 miles long, how high is the hill, to the nearest hundred feet?

8. When the angle of elevation of the sun is 75° a building casts a shadow of 125 ft. How tall is the building the nearest foot?

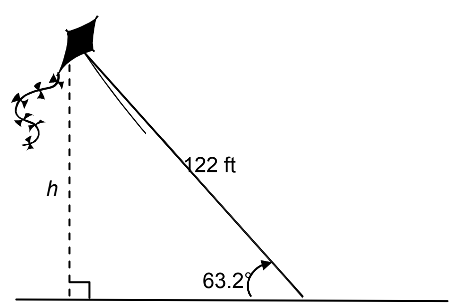
9. A 6 ft man casts a shadow that is 14 ft long. What is the angle of elevation of the sun, to the nearest degree?

10. The circle shown has a radius of r, and a center at C. If the distance DE = 36 cm, find the radius of the circle to the nearest centimeter.

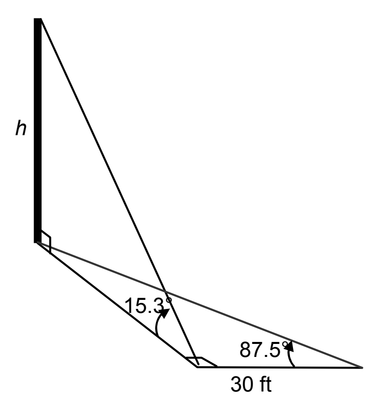




11. The cube shown has a side length of 10 cm. Find the angle formed by the diagonals AG and DG (correct to the nearest tenth of a degree.)

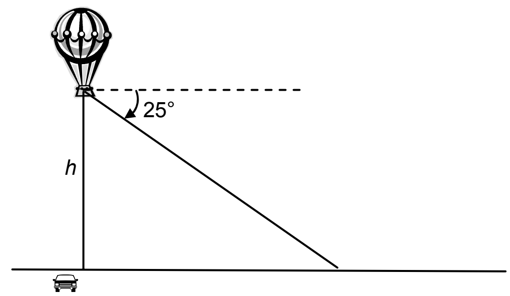


12. A kite string is extended 122 ft in length when the kite makes an angle of elevation of 63.2° with the ground. Find the altitude of the kite to the nearest foot.

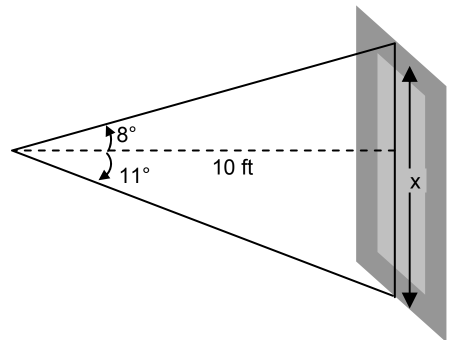


13. To measure the height of a tower across a freeway, a student takes two measurements. She stands directly across from the point at the foot of the tower, and finds that the angle of elevation to the top of the tower is 15.3°. She then walks 30 ft parallel to the freeway (at a right angle to the point at which she took the measure) and then finds that the angle from her new location to the base of the tower is 87.5°. Using this information, find the height of the tower correct to the nearest foot.

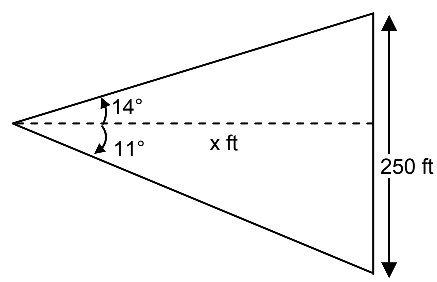
14. A hot air balloon is floating above a straight stretch of highway. To estimate how high above the ground the balloon is floating, the passengers of the balloon take measurements of a car below them. They assume that the car is traveling at 50 miles per hour. One minute after the car passes directly below the balloon they take a bearing on the car and find that the angle of depression to the car is 25°. Find the altitude of the balloon to the nearest 100 ft.



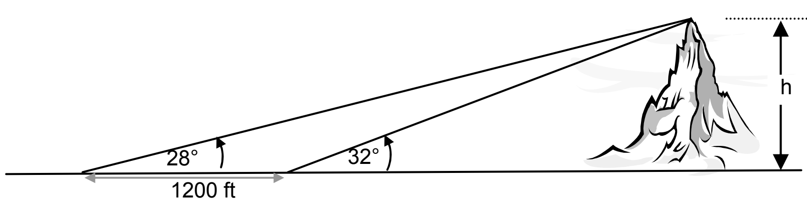
15. A man is standing 10 ft from a painting. He notices that the angle of elevation from his eyes to the top of the painting is 8° and the angle of depression to the bottom of the painting is 11°. Find the height of the painting to the nearest tenth of a foot.



16. A person standing on hill sees a tower that she knows to be 250 ft high. She observes that the angle of elevation to the top of the tower is 14°, while the angle of depression to the foot of the tower is 11°. How far is she from the tower, correct to the nearest foot?



17. To estimate the height of a particular mountain, the angle of elevation to the top of the mountain is measured to be 28°. 1200 ft closer to the mountain the angle of elevation is found to be 32°. What is the height of the mountain to the nearest hundred feet?



18. Find the dimensions of the sheet of paper needed to draw an octagon of side 12 cm, to the nearest centimeter.